MIDTERM #1 SOLUTIONS

1. $E = -\vec{\mu} \cdot \vec{B}$

(a) It is a 2-level system, $E_1 = -\mu B$; $E_2 = \mu B$, non-degenerate

$$Z = e^{+\mu B \cdot \beta} + e^{-\mu B \cdot \beta} = 2\cosh(x), \qquad x \equiv \mu B \cdot \beta = \frac{\mu B}{KT}$$

(b) $S = \frac{\bar{E}}{T} + K \ln Z$; Need to find $\bar{E} = -\frac{\partial \ln Z}{\partial \beta}$

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta} = -\frac{1}{Z} \cdot \frac{\partial Z}{\partial \beta} = -\frac{1}{Z} \cdot (\mu B) \cdot \left(e^{\mu B \cdot \beta} - e^{-\mu B \cdot \beta}\right)$$

$$\Rightarrow S = -\frac{1}{T \cdot Z} \cdot (\mu B)\left(e^{\mu B \cdot \beta} - e^{-\mu B \beta}\right) + K \ln\left(e^{\mu B \cdot \beta} + e^{-\mu B \cdot \beta}\right)$$

this is entropy of one particle, need to multiply by N to get total entropy.

(c) $$\lim_{\beta \to 0} \bar{E} = -N \cdot \mu B \frac{\left(e^{\mu B \beta} - e^{-\mu B \beta}\right)}{\left(e^{\mu B \cdot \beta} + e^{-\mu B \cdot \beta}\right)} = -N \cdot \mu B \cdot \frac{(1-1)}{2} = 0$$ ✓

(d) $$S = -\frac{K}{KT} \mu B \cdot \frac{\left(e^{\mu B \cdot \beta} - e^{-\mu B \beta}\right)}{\left(e^{\mu B \cdot \beta} + e^{-\mu B \beta}\right)} + K \ln\left(e^{\mu B \beta} + e^{-\mu B \beta}\right) \qquad (*)$$

At high temperature, expect random alignment, or $\Omega = \binom{N}{N/2} = \frac{N!}{N/2! \cdot N/2!} =$

$$= \frac{(N \cdots N/2+1)\,\cancel{N/2!}}{\cancel{N/2!}\,(N/2)!}$$ Use Stirling's formula, $N! \simeq N^N e^{-N}$

$$\frac{N^N \cdot e^{-N}}{\left((N/2)^{N/2} \cdot e^{-N/2}\right)^2} = \frac{N^N \cancel{e^{-N}}}{\cancel{e^{-N}} \cdot (N/2)^N} = \underline{2^N} \Rightarrow S = K \ln \Omega = \underline{NK \ln 2}.$$

use $(*)$ in the limit of $\beta \to 0$

$$\lim_{\beta \to 0} S = \lim_{\beta \to 0} N\left(K \cdot \beta \mu B \, \cancel{\frac{e^{\mu B \cdot \beta} - e^{-\mu B \cdot \beta}}{e^{\mu B \cdot \beta} + e^{-\mu B \cdot \beta}}} + K \ln\left(\overset{1}{e^{\mu B \cdot \beta}} + \overset{1}{e^{-\mu B \cdot \beta}}\right)\right) =$$

$$= \underline{N \cdot K \cdot \ln(2)}$$

2. $E = \frac{5}{2} RT$ $\quad (V_i, P_i) \to (V_f, P_f)$

(a) $$C_V = \frac{\partial Q}{\partial T}\bigg|_V = \frac{\partial E}{\partial T}\bigg|_V = \frac{5}{2} R$$

(b) Can use any process; use a quasi-static process at constant T, such $P \cdot V =$ constant for the process. In this case, $dE = 0$

$$\Delta S_g = \int \frac{dQ_g}{T} = \int \frac{\cancel{dE} + p\,dV}{T} = \int_{V_i}^{V_f} R \frac{dV}{V} = R \ln\left(\frac{V_f}{V_i}\right)$$

(c) $$T = \frac{P_i \cdot V_i}{R} = \frac{P_f \cdot V_f}{R} = \frac{10^6 \times 1}{8.31} = 1.2 \times 10^5 \text{ K}$$

(d) $$W_g = \int p\,dV = RT \ln\left(\frac{V_f}{V_i}\right) < 0$$, work was done on the gas to achieve compression

(e) For this, need the change in entropy of the heat bath:

$$\Delta S_{HB} = \int \frac{dQ_{HB}}{T} = \int \frac{-dQ}{T} = -R \ln\left(\frac{V_f}{V_i}\right)$$

According to the second Law of Thermodynamics,

$$\Delta S = \Delta S_g + \Delta S_{HB} = 0$$: this is a reversible process.