PH673 - High Energy Astrophysics

Assignment 5 - Sept. 5, 2007

1. *Practice with 4-vectors and tensors* .

(a) Given two 4-vectors A_i and B_i, show that $A_i B^i$ is invariant upon Lorentz transformation, i.e., $A_i B^i$ is a scalar (field).

(b) Show that the 4-velocity $u^i \equiv dx^i/ds$ is given by

$$u^i = (\gamma, \frac{\gamma}{c}\boldsymbol{v})$$

where $\boldsymbol{v}$ is the usual 3-velocity vector.

(c) Using the fact that a second-rank tensor T_{ij} is defined as the product of two 4-vectors, $T_{ij} = A_i B_j$, find the Lorentz transformation law for a second-rank tensor. [Hint: treat the T_{ij} tensor as a 4x4 matrix.]

(d) Using the results from (c), show that the η_{ij} tensor is invariant upon Lorentz transformation, i.e., $\eta_{ij} = \eta'_{ij}$. You may do so by checking at least the η_{00} term.

2. *Relativistically invariant quantities.*

(a) Consider a set of N particles with relativistic distribution function $f(p^i)$, i.e.,

$$N = \int d^4 p f(p^i) \delta(p_a p^a - m^2 c^2)$$

in which the $\delta(p_a p^a - m^2 c^2)$ term ensures that all particles have mass m.

Use the following information:

(1) $p^0 = \gamma mc > 0$, which is not explicitly accounted for in the above equation;

(2) Assume that both $d^4 p$ and N are relativistic invariants;

(3) The Dirac delta function satisfies the following property:

$$\delta(\phi(x)) = \frac{1}{|\dot{\phi}(a_i)|} \delta(x - x_i)$$

in which a_i are the roots of the function $\phi(x) = 0$, and summation in implied over all roots;

to show that $d^3 \boldsymbol{p}/E$ is a relativistic invariant.

(b) Starting with $E = \gamma mc^2$, show that $E \cdot d^3 \boldsymbol{x}$ is a relativistic invariant (scalar).

(c) Finally, conclude that $d^3 \boldsymbol{p} \cdot d^3 \boldsymbol{x}$ is a relativistic invariant, although each component individually is not.