RADIATION REACTION

Conservation of energy arguments require that the energy radiated by the accelerated particles causes a deceleration of the charged particles. Therefore, one expects a drag force $\vec{F}$ on them, given by

$$P = \langle \vec{f}\cdot\vec{v}\rangle = \langle \frac{d\mathcal{E}}{dt}\rangle = \langle -\frac{2q^2a^2}{3c^3}\rangle \qquad (1)$$

where the $\langle\ \rangle$ indicates suitable time averages, and consider the case of single particles of charge q, and have used the Larmor formula.
When averaging the acceleration $\vec{a}$ over a time T, can put:

$$\langle a^2\rangle = \frac{1}{T}\int_0^T a^2 dt = \frac{1}{T}\int_0^T dt\,(\dot{\vec{v}}\cdot\dot{\vec{v}}) = \frac{1}{T}\int_0^T dt\left[\frac{d}{dt}(\vec{v}\cdot\dot{\vec{v}}) - \vec{v}\cdot\ddot{\vec{v}}\right] =$$

$$= \frac{1}{T}(\dot{\vec{v}}\cdot\vec{v})\Big|_0^T - \frac{1}{T}\int \vec{v}\cdot\ddot{\vec{v}}\,dt$$

In the limit of $T\to\infty$, the first term will vanish for a bounded motion. The second term can be re-written as:

$$\langle a^2\rangle \underset{T\to\infty}{\longrightarrow} -\frac{1}{T}\int \ddot{\vec{v}}\cdot\vec{v}\cdot dt \qquad (2)$$

Putting together equations (1) and (2) obtain that:

$$\frac{1}{T}\int \vec{f}\cdot\vec{v}\,dt = +\left(\frac{1}{T}\int \ddot{\vec{v}}\cdot\vec{v}\cdot dt\right)\times\frac{2q^2}{3c^3}$$

$$\Rightarrow \quad \vec{f}_{drag} = \frac{2q^2}{3c^3}\cdot\ddot{\vec{v}} \qquad (3)$$

Although applications of equation (3) can be complicated, one important question to answer is: when is the drag force comparable to the Lorentz force caused by an EM field?

$$\vec{F}_L = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B} = m\cdot\vec{a}$$

$$\Rightarrow \vec{f}_{drag} = \frac{2q^3}{3c^3m}\cdot\left(\dot{\vec{E}} + \frac{d}{dt}\left(\frac{\vec{v}\times\vec{B}}{c}\right)\right) = \frac{2}{3}\frac{q^3}{c^3}\frac{1}{m}\left(\dot{\vec{E}} + \frac{\vec{a}}{c}\times\vec{B} + \frac{1}{c}\vec{v}\times\dot{\vec{B}}\right) =$$

$$= \frac{2}{3}\frac{q^3}{c^3}\frac{1}{m}\left(\dot{\vec{E}} + \frac{1}{mc}\left(q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}\right)\times\vec{B} + \frac{1}{c}\vec{v}\times\dot{\vec{B}}\right) =$$

$$= \frac{2}{3}\frac{q^3}{c^3 m}\left(\dot{\vec{E}} + \frac{q}{mc}\vec{E}\times\vec{B}\right) + O\left(\frac{\vec{v}}{c}\right)$$

Considering non-relativistic approximation, can neglect $\vec{v}/c$ terms.

$$\Rightarrow \vec{f}_{drag} \simeq \frac{2}{3}\frac{q^3}{c^3 m}\left(\dot{\vec{E}} + \frac{q}{cm}\vec{E}\times\vec{B}\right)$$

Also, assume the plane-wave solution of the EM field,

$$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}, \quad \vec{B} = \vec{n}\times\vec{E} \qquad \left(|\vec{B}| = |\vec{E}|\right)$$

and the instantaneous rest frame of the charged particle is considered. In this case, the condition that $\vec{F}_{drag} \ll \vec{F}_L$ is:

$$q\vec{E} \gg \frac{2}{3}\frac{q^3}{c^3 m}\left(i\omega\vec{E} + \frac{q}{cm}E^2\vec{n}\right) \qquad (\vec{v} = 0)$$

$$\Rightarrow \begin{cases} qE \gg \dfrac{2}{3}\dfrac{q^3}{c^3 m}\cdot\omega E \\ \text{— and —} \\ qB \gg \dfrac{2}{3}\dfrac{q^3}{c^3 m}\cdot\dfrac{q}{cm}\cdot B^2 \end{cases}$$

must both be satisfied

$$\Rightarrow \begin{cases} \omega \ll \dfrac{mc^3}{q^3} \\ B \ll \dfrac{c^4 m^2}{q^3} \end{cases} \qquad (1)$$

Conditions (1) can be considered as the limits of applicability of the classical electrodynamics.

Ex: for $q = q_e$, $m = m_e$,

$$\omega \ll \frac{(9.1\times10^{-28})\times(3\times10^{10})^3}{(4.8\times10^{-10})^3} = \frac{2.2\times10^{32}}{2\pi}\ \text{Hz} \quad \text{or} \quad \lambda = \frac{3\times10^{10}}{3.5\times10^{31}} = 10^{-21}\ \text{cm} \quad (\sim 10^{-13}\ \text{Å})$$

$$B \ll \frac{(3\times10^{10})^4\times(9.1\times10^{-28})^2}{(4.8\times10^{-10})^3} \simeq 6.1\times10^{15}\ \text{gauss}$$