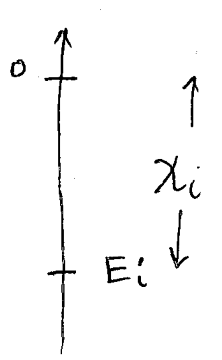
The Saha Equation

The Saha equation aims to provide a description of the absolute number of atoms in different ionization stages, e.g., atoms that have lost a given number of electrons and are now positively charged. In order to make further progress, we need to consider an additional consideration, which is the "degeneracy" of the "statistical weight" of a given energy level. In other words, a number g_i of different quantum states can give rise to the same energy state, and therefore an energy state or level "weighs" more according to its value of g_i. Accordingly, a more accurate statement of Boltzmann's hypothesis is the Boltzmann distribution:

$$\frac{N_A}{N_B} = \frac{g_A}{g_B} e^{-(E_A - E_B)/kT} \qquad (3)$$

which takes into proper account the statistical weights of each energy level; g_A and g_B are calculated by quantum mechanics.

When considering ions in two ionization stages, N_i and N_{i+1}, for simplicity (neighboring levels of ionization), the Boltzmann distribution is determined by the probability of one additional electron to become free; assume for simplicity that the two ions/atoms are in their ground levels, of degeneracy g_i and g_{i+1} (this assumption will later be relaxed)



χ_i is the ionization energy from state i

In ionization stage i, the atom in its ground state (or electron) has an energy E_i; in ionization stage $(i+1)$, the electron will have acquired an extra energy of:

$$E = \chi_i + p^2/2m \qquad , p \geq 0 \text{ is momentum}$$

since χ_i is required for ionization, and $p^2/2m$ is the additional kinetic energy of the (now free) electron.

Application of Boltzmann's distribution to the transition between i-th ionization state and $(i+1)$th state is:

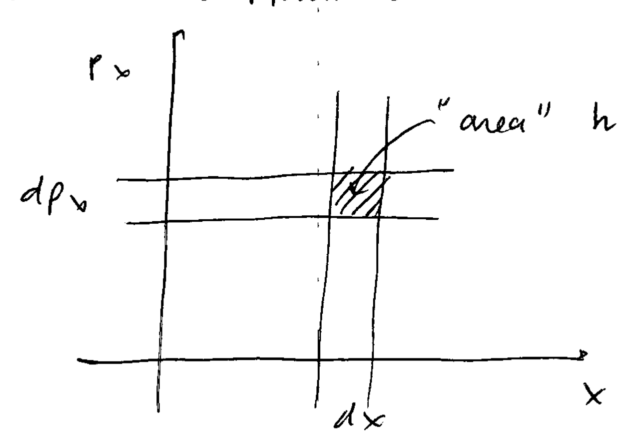
$$\frac{N_{i+1}}{N_i} = \frac{g_{i+1}}{g_i} \int g_e(p)\, e^{-(\chi_i + p^2/2m)/KT} \qquad (4)$$

where $g_e(p)$ is the degeneracy, or number of "levels", available to an electron of momentum p. From a classical point of view, there is no such thing as a "classical level", since levels are a quantum physical concept.

The calculation of $g_e(p)$ is performed by use of Heisenberg's uncertainty principle, which states that the uncertainty in the knowledge of momentum and position is such that

$$dp_x \, dx = h$$

where h is Planck's constant.



This view affords the concept that a volume of h^3 is the infinitesimal cell of phase-space, which can be called a unique "state"; this number is to be further multiplied by a factor of 2, to account for two possible spin states of the electron. Therefore for an electron occupying a physical volume V_e, the number of free phase-space cells available for a momentum range $p, p+dp$ is:

$$g_e(p) = \frac{V_e \, 4\pi p^2 \, dp}{h^3} \qquad (5)$$

We can now return to equation (4) and perform the integral:

$$g_e = \left(\frac{V_e}{h^3} \int_0^\infty g_e(p) \, e^{-p^2/2mkT} \, 4\pi p^2 \, dp \right) \cdot 2$$

The integral can be solved by change of variables,

$$x = p^2/2mkT \quad , \quad dx = 2p\,dp/2mkT$$

$$\Rightarrow g_e = \frac{2V_e}{h^3} \int_0^\infty e^{-x} \, 4\pi \cdot 2mkT \, \frac{dx}{2p} \, 2mkT =$$

$$= \frac{8\pi V_e}{h^3} \int_0^\infty e^{-x} \frac{x\,dx}{\sqrt{2mkT\,x}} \frac{(2mkT)^2}{2} =$$

$$= \frac{4\pi V_e}{h^3} (2mkT)^{3/2} \int_0^\infty e^{-x} x^{1/2} \, dx .$$

Recall that $\int_0^\infty e^{-x} x^{1/2} dx = \sqrt{\pi}/2$

$$\Rightarrow g_e = \frac{4\pi V_e}{h^3} (2mkT)^{3/2} \frac{\sqrt{\pi}}{2} = \frac{2\,(2\pi m kT)^{3/2}}{m_e k h^3} \qquad (6)$$

and therefore we can write equation (4), using $n_e = 1/V_e$ as the electron number density

$$\Rightarrow \frac{n_{i+1} \cdot n_e}{n_i} = \frac{(2\pi m k T)^{3/2}}{h^3} \frac{2 g_{i+1}}{g_i} e^{-\chi_i / kT} \qquad (7)$$

which is known as <u>Saha's Equation</u>; the equation depends on the number of free electrons, since they are generated in the ionization process.

<u>Interpretation of Balmer lines from stellar spectra</u>

We are now in a position to understand one of the key features of stellar spectra, which is the fact that H Balmer lines peak at about ~~k~~$T = 10{,}000$ K.

In order for having strong $H\alpha, H\beta$ etc. lines, two conditions must be satisfied:

1) H atoms must be primarily in its <u>first excited state</u> of energy $E_2 = -3.4\,eV$
2) H atoms must also be <u>neutral</u>, that is, they cannot be ionized.

	N_1 :	number of atoms in	$n=1$ ~~ground~~ state
Call	N_2 :	number of atoms in	$n=2$ (excited) state
	N_I :	" "	that are neutral
	N_{II} :	" "	that are ionized.

we are interested in finding <u>N_2/N_{total}</u>, where both N_1 and N_2 are particular cases of N_I.

$$\Rightarrow \frac{N_2}{N_{tot}} = \frac{N_2}{N_1 + N_2 + \sum_{j=3}^{40} N_j + N_{II}} = \frac{N_2}{N_I + N_{II}}$$

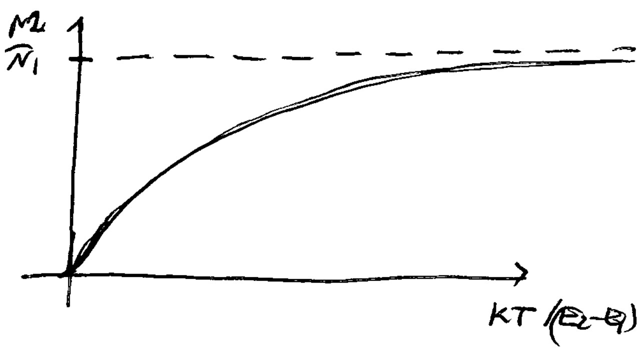
We can also assume that $N_I + N_2 \simeq N_I$, meaning that the most common states of the neutral atom are $n=1$ (the ground state) and $n=2$ (the first excited state)

$$\Rightarrow \frac{N_2}{N_{tot}} \simeq \frac{N_2}{N_{tot}} \cdot \left(\frac{N_I}{N_1 + N_2} \right) = \frac{N_2}{N_1 + N_2} \cdot \frac{N_I}{N_{tot}} =$$

$$= \frac{N_2/N_1}{1 + N_2/N_1} \cdot \left(\frac{1}{1 + N_{II}/N_I} \right) \qquad (8)$$

which we can now evaluate, since N_2/N_1 is described by the Boltzmann distribution, and N_{II}/N_I by the Saha equation.

1) $$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\overbrace{(E_2-E_1)}^{\text{positive}}/kT}$$



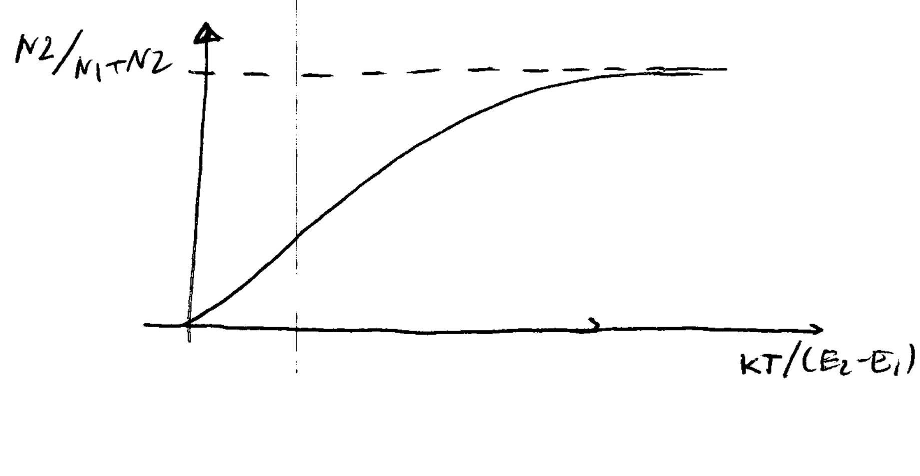
$E_2 = -3.4\ eV$; $E_1 = -13.6\ eV$

$\Rightarrow E_2 - E_1 = +10.2\ eV$

$$\lim_{kT \to 0} \frac{N_2}{N_1} = 0 \quad ; \quad \lim_{kT \to \infty} \frac{N_2}{N_1} = 1 \cdot \frac{g_2}{g_1}$$

Therefore the term $N_2/N_1 / (1 + N_2/N_1)$, or $N_2/(N_1+N_2)$ is

~~[illegible] Saha's equation~~



Notice that $(E_2 - E_1) = 10.2\ eV$, which corresponds to a temperature of:

$$kT = (E_2 - E_1) \Rightarrow T = \frac{10.2\ eV}{8.61\ eV/k} = 110,000\ °K$$

For $T << 110,000\ °K$, most H atoms are in the ground state according to the Boltzmann distribution.

2) Using Saha's equation:

$$\frac{N_{II}}{N_I} = \frac{(2\pi m_e kT)^{3/2}}{n_e h^3} \frac{2 g_2}{g_1} e^{-\chi_1/kT} \qquad (9)$$

First, consider that g_2 corresponds to the case of a ionized H atom, which is just a proton, and no electron - therefore $g_2 = 1$. Assuming that the neutral H atom is mainly in the ground state, $g_1 = 2$, as observed earlier. Also,

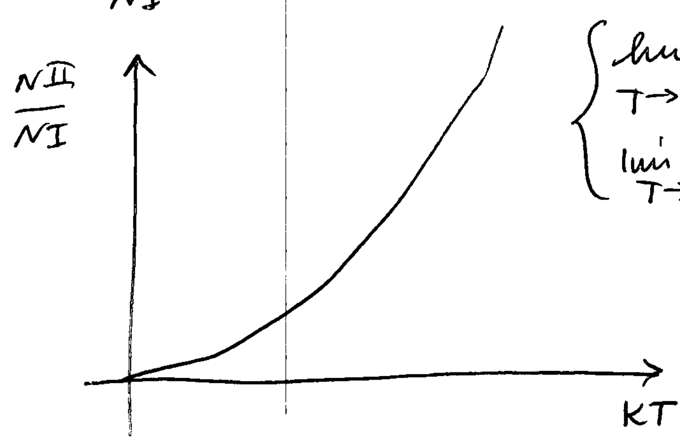
$$\chi_1 = 13.6\ eV$$

A possible complication in Saha's equation is the term "n_e" which appears on the right-hand side of (9). Given that the free electrons may obey the ideal gas law, especially at low pressure,

$$P_e = n_e \cdot kT$$

where P_e is the electron pressure. Assuming a known value of the pressure, equation (8) has the behavior of

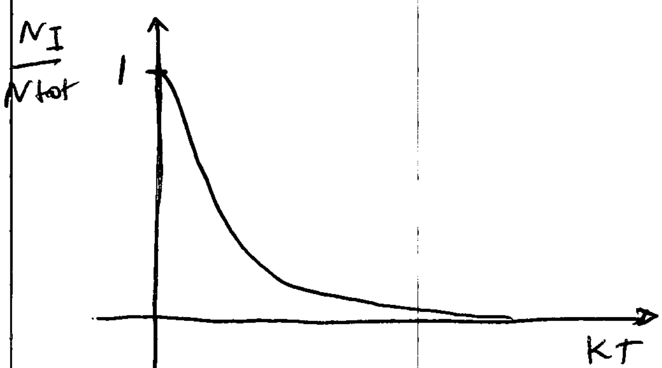
$$\frac{N_{II}}{N_I} \propto T^{5/2} e^{-\chi_1/kT}$$



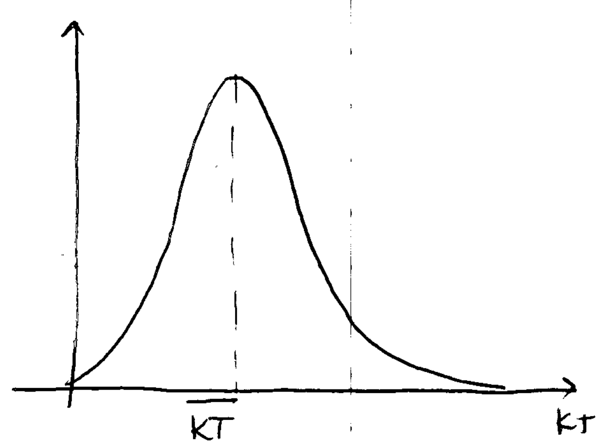
$$\begin{cases} \lim\limits_{T \to 0} \frac{N_{II}}{N_I} = 0 \\ \lim\limits_{T \to \infty} \frac{N_{II}}{N_I} = \infty \end{cases}$$

The function of interest in equation (8) is:

$$\frac{1}{1 + N_{II}/N_I} = \frac{N_I}{N_{tot}}$$



The combined behavior of N_2/N_{tot} therefore displays a peak.



The peak can be obtained analytically by taking the derivative of equation (8), and it is found at $\underline{KT \simeq 10{,}000\,K}$, which is in excellent agreement with the fact that A0 stars have the strongest Balmer lines. At that temperature, we can calculate N_{II}/N_I from Saha's equation assuming a characteristic pressure of $P_e = 20\,Pa$, or $P_e \simeq 2\times10^{-4}$ atm, which is a typical value for stellar atmospheres where the absorption lines originate.

$$\Rightarrow N_{II}/N_I \simeq 1 \quad \text{, as expected.}$$