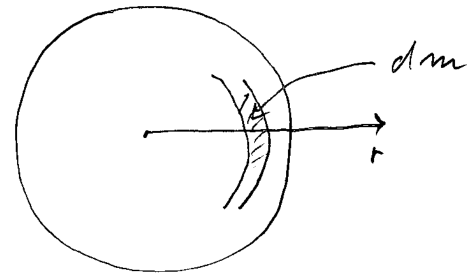
Hydrostatic equilibrium and stellar interiors

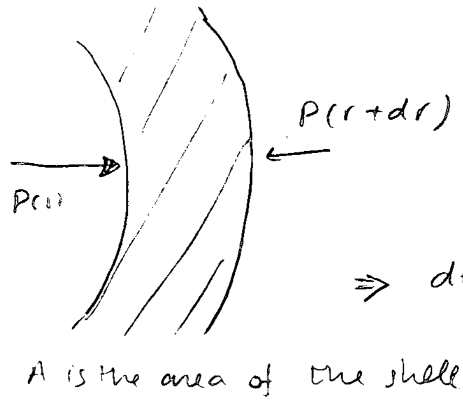
An important feature of stellar interiors, and many other astrophysical systems, is that there is a balance between the force of gravity and the pressure of the gas.
Consider a self-gravitating spherical distribution of gas



Each shell between radii $r, r+dr$ is subject to two forces: the force of gravity

$$dF_G = -\frac{GM(r)\,dm}{r^2} \qquad (1)$$

where $M(r)$ is mass within radius r, and the force due to the pressure of the gas itself. In order to evaluate this term, must consider that $P(r)$ is, in general, a radial function which decreases towards large radii. The net pressure force is therefore given by



$$dP = P(r) - P(r+dr)$$

$$\Rightarrow dF_P = dP \cdot A \qquad (2)$$

where A is the area of the shell (or of any section thereof).

$$\Rightarrow -\frac{GM(r)\,dm}{r^2} = dP \cdot A$$

where dm is the mass of the shell (or of any section thereof)

$$\Rightarrow \quad dm = \rho \, dV = \rho \cdot A \cdot dr$$

Therefore the balance between the two forces is expressed as.

$$-\frac{G M(r) \, \rho \cancel{A} \cdot dr}{r^2} = dP \cdot \cancel{A}$$

$$\Rightarrow \quad \frac{1}{\rho} \frac{dP}{dr} = -\frac{G M(r)}{r^2} \qquad (3)$$

which is known as the equation of hydrostatic equilibrium, which applies to any spherically symmetrical distribution of gas. As remarked earlier, Eq. (3) can only be satisfied is

$$\frac{dP}{dr} < 0$$

meaning that self-gravity will lead to a radial decrease of the pressure, which is required to balance the pull of gravity.

<u>Estimate of pressure at center of Sun</u>

Assuming that the Sun is in hydrostatic equilibrium, Eq. (3) affords a simple way to estimate the pressure at its center.
Assuming a uniform density.

$$\frac{dP}{dr} \approx -\frac{P_c}{R_\odot} = -\frac{G M_\odot}{R_\odot^2} \cdot \rho_\odot \qquad , \rho_\odot \simeq 1 \text{ g/cm}^3$$

$$\Rightarrow \quad P_c = \frac{G M_\odot \rho_\odot}{R_\odot} \simeq \frac{6.67 \times 10^{-8} \times 2 \times 10^{33}}{6.5 \times 10^{10}} \simeq 2 \times 10^{14} \text{ Pa} \simeq \underline{\underline{10^{9} \text{ atm}}}$$

The Kelvin-Helmholtz Timescale

Until the 1900's it was not understood what the source of stellar (Solar, in particular) luminosity was. One opinion was that the gravitational contraction of the Sun would be sufficient to supply the observed luminosity.

The reasoning was based upon the Virial theorem, which states that a bound system:

$$-2\langle K\rangle = \langle U\rangle$$

and therefore the total energy is

$$E = \langle U\rangle + \langle K\rangle = +\frac{1}{2}\langle U\rangle < 0 \qquad (3)$$

Equation (3) states that when there is a change ΔU in gravitational potential energy, for example because of collapse, then $\frac{1}{2}\Delta U$ goes into the total energy ΔE, and the other half is available to be radiated away. From this was born the theory that the Sun radiates via gravitational energy.

We can estimate the viability of the theory by considering

$$U_G = -\int \frac{G M(r)\, dm}{r} = -\int_0^R \frac{G M(r)\rho\, 4\pi r^2 dr}{r}$$

where R is the star's radius; assume for simplicity that

$$\rho \sim \bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}, \quad M(r) = M\cdot\frac{r^3}{R^3}$$

$$\Rightarrow U_G = -\int_0^R G M \frac{r^3}{R^3}\cdot\frac{M}{\frac{4}{3}\pi R^3}\,\frac{4\pi r^2 dr}{r} = -\frac{3 G M^2}{R^6}\left.\frac{r^5}{5}\right|_0^R =$$

$$= -\frac{3}{5} G\frac{M^2}{R} \quad (4) \qquad \text{and} \qquad E = -\frac{3}{10} G\frac{M^2}{R} \quad (4')$$

If the Sun emits via gravitational collapse, then one would assume that the Sun had an energy available for radiation, since its origin, of

$$\Delta E = (E_f - E_i) = -\frac{3}{10}\frac{G M_\odot^2}{R_\odot} \simeq 10^{41}\ J$$

assuming that $E_i = 0$, when Sun initially formed. Assuming that the Sun had the same luminosity $L_\odot \simeq 4\times10^{26}\ W$, then this process would have supplied energy for a period of

$$t = \frac{E}{L} = \frac{10^{41}\ J}{4\times10^{26}\ J/s} = 2.5\times10^{14}\ s \quad \text{or} \quad \underline{\underline{10^{7}\ \text{years}}}$$

which is much shorter than the age of the Sun, which is estimated at $\simeq 10^{10}$ years at least. This calculation showed that the source of energy in the Sun could not be gravitational collapse.

Thermodynamics of stellar interiors

The first law of thermodynamics states that

$$dQ = dE + dW \quad , \quad dW = p\,dV \qquad (1)$$

where dQ is heat supplied to a given volume element, dE is the change in internal energy, and dW is the work done by the element onto its surroundings.
Stellar interiors can be approximated as adiabatic systems if the source of energy at the center is not taken into account.
In the case of an adiabatic and ideal gas,

$$\begin{cases} dE = -p\,dV & (2) \\ pV = \nu RT & , \ \nu \text{ is number of moles} \end{cases}$$

$$\Rightarrow d(PV) = P\,dV + V\,dp = \nu R\,dT \qquad (2')$$

For a monatomic ideal gas, $E = \frac{3}{2}\nu RT$ is the energy possessed by the entire system.
Also, can define specific heats at constant pressure and at constant volume of the gas as:

$$c_V = \frac{dQ}{dT}\Big|_V \quad ; \quad c_p = \frac{dQ}{dT}\Big|_p$$

which, for a monoatomic ideal gas, are given by

$$\begin{cases} c_V = \dfrac{d(\frac{3}{2}\nu RT)}{dt} = \dfrac{3}{2}\nu R \\ c_p = \dfrac{d}{dt}\Big(\frac{3}{2}\nu RT + \underbrace{pV}_{= nRT}\Big)\Big|_p = \frac{3}{2}\nu R + \nu R = c_V + \nu R \end{cases} \qquad (3)$$

and also $E = c_V \cdot T$ or $dE = c_V\,dT$

Return now to the adiabatic process described by (2), and use the expressions obtained for c_p and c_v

$$\Rightarrow P dV + V dP = \nu R\, dT = \nu R \cdot \frac{dE}{c_V} =$$

$$= \nu R \left(- \frac{P dV}{c_V} \right)$$

which is a differential equation for P and V, and therefore can in principle be solved for one quantity as function of the other. The integration can in fact be performed by following these steps

$$dP \cdot V = - dV \left(\frac{\nu R P}{c_V} + P \right) = - dV \left(\frac{\nu R}{c_V} + 1 \right) P$$

$$= - dV \left(\frac{c_V + \nu R}{c_V} \right) P = - dV \cdot \frac{c_P}{c_V} P$$

call $\gamma \equiv c_P / c_V$ the ratio of specific heats, $\gamma > 1$

$$\Rightarrow \frac{dP}{P} = - \gamma \frac{dV}{V}$$

which can be integrated to obtain

$$\ln P \propto -\gamma \ln V \quad \text{or} \quad \underline{P V^{\gamma} = \text{constant}} \quad (4)$$

which is known as the adiabatic relationship. This can also be transformed, by using the ideal gas law $PV = \nu R T$ $\left(V = \frac{\nu R T}{P} \right)$

$$\Rightarrow \begin{cases} T V^{\gamma - 1} = \text{const} , & (4') \\ P \left(\frac{T}{P} \right)^{\gamma} = \text{const} , \text{ or } P^{1-\gamma} \cdot T^{\gamma} = \text{const} , \text{ or } T \cdot P^{1-\gamma/\gamma} = \text{const}. \end{cases}$$