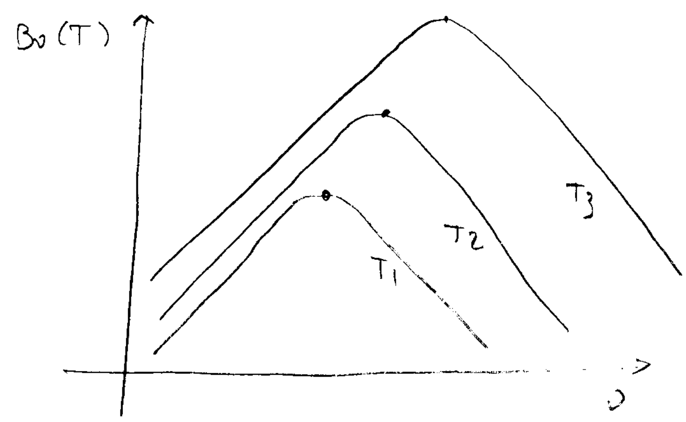
Properties of black-body radiation

1) The Wien displacement law states that the wavelength of peak emission λ_{max} is related to the temperature as:

$$\lambda_{MAX} T = 0.289 \text{ cm K}$$

(it can be proven via a derivative of the Black-body intensity)

Since $\nu = 2\pi/\lambda$, it also states that the peak frequency increases with temperature



$T_1 < T_2 < T_3$

2) The intensity is such that

$$B_\nu(T_1) < B_\nu(T_2) \quad \text{if } T_1 < T_2, \quad \forall \nu \in \mathbb{R}$$

In other words, the intensity is a monotonic function of temperature, at all frequencies, implying that the curves of increasing T are "nested" into one another (as in figure).

3) The Stefan-Boltzmann equation is the finding that the intensity, integrated over all frequencies, is proportional to T^4:

$$I = \int_0^\infty I_\nu \, d\nu = \int_0^\infty \frac{2\pi\nu^3 h}{c^2} \left(\frac{1}{e^{h\nu/KT} - 1} \right) d\nu \quad ; \text{ call } x = \frac{h\nu}{KT}$$

$$I = \frac{\sigma}{\pi} T^4 \quad (6) \quad [ergs/s\,cm^2\,srad], \quad \sigma = \frac{2\pi^5 k^4}{15 c^2 h^3}$$

where $\sigma = 5.67 \times 10^{-8}\ W\ m^{-2}\ K^{-4}$ is the Stefan-Boltzmann constant.

Application of black-body radiation to stellar emission

Equation (6) is the tool necessary to link observations of stars with the star's physical parameters.

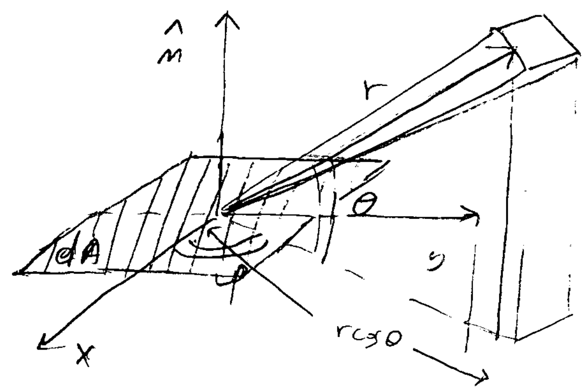
Assume a star of radius R and of temperature T, the surface area is given by $A = 4\pi R^2$, and therefore the total luminosity emitted is the integral:

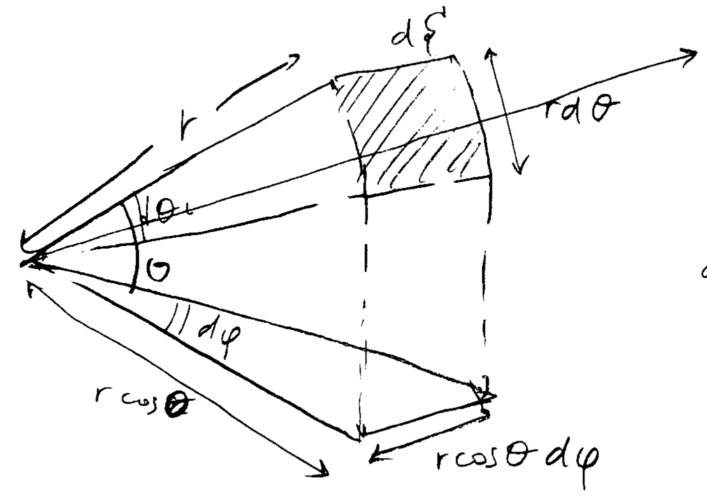
$$L = \int_A dA \int_\Omega d\Omega \int I_\nu \, d\nu =$$

$$= I \int_\Omega d\Omega \int_A dA ,$$

The integration over area gives the obvious contribution $\int_A dA = A = 4\pi R^2$, while the integration over solid angle must be handled with more care.

Use spherical coordinates r, θ and φ centered at the element of area dA. For a given direction (θ, φ), the area dA has a projection $dA \cdot \sin\theta$ along that direction (from direction θ, φ, area dA is seen to have a perpendicular extent $dA \sin\theta$)





$$d\Omega = \frac{d\xi}{r^2}$$

The surface element $d\xi$ perpendicular to the direction of the emitted radiation is, in spherical coordinates

$$d\xi = r^2 \cos\theta \, d\theta \, d\varphi \quad , \quad \begin{cases} \theta \in [0, \pi/2] \\ \varphi \in [0, 2\pi] \end{cases}$$

$$\Rightarrow d\Omega = \cos\theta \, d\theta \, d\varphi$$

Therefore we need to integrate with respect to all possible direction θ, φ

$$\Rightarrow L = I \cdot \int_A dA \int \sin\theta \cdot \cos\theta \, d\theta \, d\varphi =$$

$$= I \cdot A \cdot \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

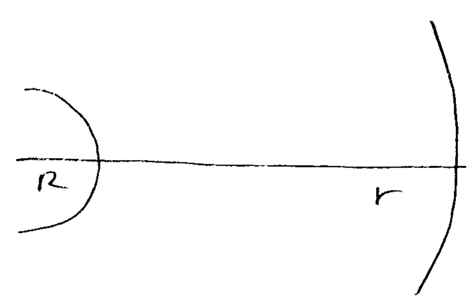
Notice that $\int_0^{\pi/2} \sin\theta\cos\theta \, d\theta = \frac{1}{2} \sin^2\theta \Big|_0^{\pi/2} = \frac{1}{2}$

$$\Rightarrow \quad L = I \cdot A \cdot \left(2\pi \, \frac{1}{2}\right) = \pi A \cdot I$$

Using equation (6), this becomes:

$$\boxed{L = \sigma \cdot A T^4 = 4\pi R^2 \sigma T^4} \qquad (7) \qquad [\text{erg/s}]$$

Notice that the primary observable is not the luminosity, which implies a detection of all light emitted by the star. The main observable is the flux, or power detected per unit area at a given distance of the observer, r



$$F = \frac{L}{4\pi r^2} = \sigma T^4 \left(\frac{R}{r}\right)^2 \quad \left[\frac{\text{erg}}{\text{s cm}^2}\right]$$

Equation (7) is a fundamental tool for measuring a star's radius. In fact, from the spectral shape of the continuum emission, T can be estimated first; then, knowing the distance to source, from the flux one can observe the luminosity, and thus the radius.