CHAPTER 13
GRAVITATION

1. Newton's Law of gravitation

Two objects of mass m and M, separated by a distance r, experience a mututal force expressed by

$$\boldsymbol{F_G} = G \cdot \frac{m \cdot M}{r^2} \hat{\boldsymbol{r}} \tag{1}$$

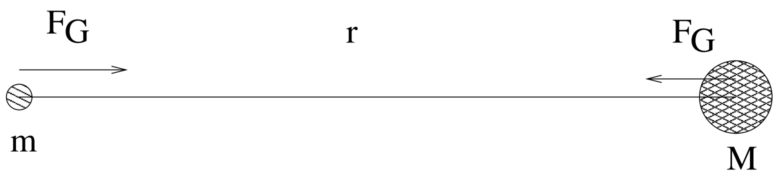

Fig. 1.— Gravitational force between two massive objects

where $\hat{\boldsymbol{r}}$ indicates the direction that connects the two objects. G is a constant know as *gravitational constant*, $G = 6.67 \times 10^{-11}$ [N m^2 kg^{-2}]. Both objects experience a force with *same* magnitude and opposite direction . This law was determined experimentally over the course of several centuries of observations of celestial bodies.

The Law of Gravity holds true for *any* two objects that have a mass. The presence of a gravitational acceleration that pulls all objects towards the center of the Earth is a special case of Equation 1. The gravitational force given by Equation 1 is conservative, i.e., its work depends only on position, not on path. This fact will be used to define a gravitational potential energy U_G, which generalizes the early definition which holds true only in the vicinity of the Earth's surface ($U_G = m \cdot g \cdot h$).

Given that the radius of the earth is $R_E \simeq 6.4 \times 10^6$ [m], $M_E \simeq 6 \times 10^{24}$ [kg], any objects on the surface of the Earth will experience an acceleration of:

$a_G = \frac{F_G}{m} = \frac{G \cdot M_E}{R_E} = \frac{6.67 \cdot 10^{-11} \times 6 \cdot 10^{24}}{(6.4 \cdot 10^6)^2} = 9.8$ [m s^{-2}],

which is, in fact, the usual value of the gravitational acceleration of an object near the surface of the Earth. It is clear that a_G varies according to altitude, and it decreases rapidly for objects that move away from Earth, e.g., the Space Shuttle or a rocket.

Since the direction of $\boldsymbol{F_G}$ is that along the line connecting the two bodies, its magnitude

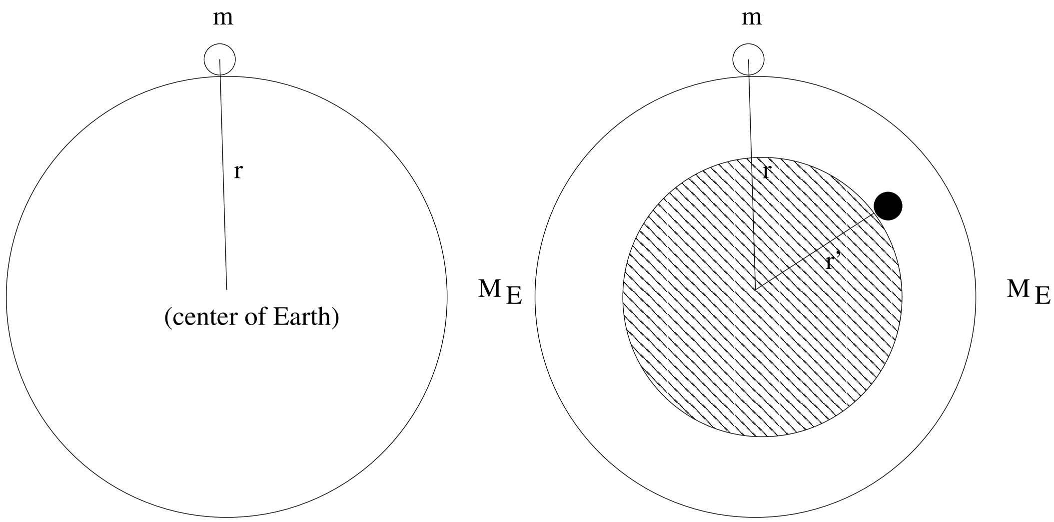

Fig. 2.— Gravitation near the surface of the Earth (left), and of an object inside the Earth (right).

is given by

$$F_G = -G\frac{M \cdot m}{r^2} \quad (2)$$

where the negative sign bears out the fact that the direction of $\boldsymbol{F_G}$ and $\boldsymbol{r}$ are opposite. The negative sign will be essential when deriving the potential energy U_G.

1.1. Gravitational force *inside* the Earth

In the calculation leading to $a_G = 9.8$ [m s^{-2}] we assumed that all of the Earth's mass is concentrated at its center. This property can be shown rigorously using the tools of calculus. It can also be shown that, for an object *inside* the Earth (at radius $r' < R_E$), than the only mass that causes the attraction is that of a sphere *within* the object's position:

$a_G(r') = \frac{G \cdot M(r')}{r'^2}$

where $M(r')$ is the Earth's mass within radius r', $M(r') < M_E$. This result can be understood with reference to Figure 2. The shell between the object's position and the Earth's surface attract the object in different direction, giving rise to a net null force. The sphere of radius r', on the other hand, yields a net force that is directed towards the center of the Earth. This result applies to any object: the only mass M that is relevant to attract an object of mass m is that *within* the radius at which m is located.

1.2. Effects of Earth's rotation

As the Earth rotates, any object on its surface will experience a centripetal force. Since

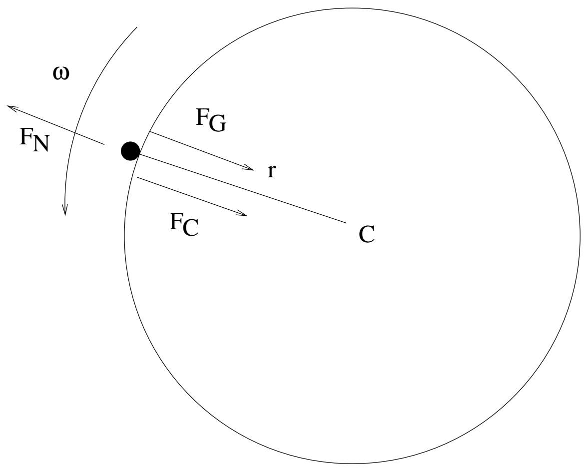

Fig. 3.— Overhead view of the Earth's rotation

$\omega = \frac{2\pi}{24}\left[\frac{\text{rad}}{\text{h}}\right] = 7.2 \times 10^{-5}\left[\frac{\text{rad}}{\text{s}}\right]$, the centripetal acceleration near the Earth's surface is $a_C = \omega^2 \cdot R_E = 3.3 \cdot 10^{-2}$ [m s^{-2}], small compared to g.

The rotation has the following effect. A person's weight is measured as the normal reaction F_N; say, when one uses a scale, its reading equals the magnitude of the Earth's normal reaction, since the scale reacts to the person's actual weight. In addition to the person's actual weight, the scale also considers the rotation of the Earth, since

$$F_N - F_G = -F_C \Rightarrow F_N = F_G - F_C$$

Although the correction is small, the rotation causes a reduction of F_N with respect to the case in which there is no rotation, i.e., the scale's reading is somewhat smaller than the person's actual weight, as defined by F_G.

2. Gravitational potential energy

The most important feature of the gravitational force of Equation 1 is that the force varies with distance, in particular it drops as $\frac{1}{r^2}$. The associated potential energy can no longer be simply proportional to the height h ($U_G = mg \cdot h$). The gravitational potential energy is now given according to the following relationship:

$dU_G = -dW = -F \cdot dr$, or $F = -\frac{dU_G}{dr}$.

Since $F = G\frac{M \cdot m}{r^2}$, for all points $r > R_E$, the gravitational potential is given by the following integration:

$\Delta U_G = -\int F_G \cdot dr = -\int -G\frac{M \cdot m}{r^2}dr = -\ G\frac{M \cdot m}{r}\Big|_{in}^{fin} = -G \cdot \frac{M \cdot m}{r_{fin}} + G \cdot \frac{M \cdot m}{r_{in}}$

It is convenient to fix as reference value for the potential energy the point $r = \infty$, which equals to$U(\infty) = 0$: the value of the gravitational potential energy of an object at an infinite distance from another massive object is null. In this case,

$$U_G = -\frac{G \cdot M \cdot m}{r} \tag{3}$$

and the gravitational potential energy is a negative number. Unlike the case of gravity near the surface of the Earth, in which we have the freedom to change the reference point for U_G in each situation, here we fix the reference point once and for all – in a sense, this is a useful simplification.

Example: Calculate the work done by the gravitational force in bringing an object of mass m from an infinite distance to a distance of R from the Earth. Assume that $R > R_E$, where R_E is the Earth radius.

With reference to Figure 1, it is clear that the mass m is naturally attracted by M, so that the force $\boldsymbol{F_G}$ on m is directed towards M, and the work done by $\boldsymbol{F_G}$ will be positive. Also, consider that $m << M$, so that the mass M does not move significantly.

$W = \int_{\infty}^{R} -G\frac{Mm}{r^2}dr = G\frac{Mm}{r}|_{\infty}^{R} = G \cdot Mm(\frac{1}{R} - \frac{1}{\infty}) = -(U_G(R) - U_G(\infty)) = -\Delta U_G$

It is important to remark that, as usual, it is only the change in potential energy that matters, and that the change in potential energy equals the negative of the work done by that force, as shown above.

Exercise 13.1: Find the work done by the gravitational force of a satellite ($m = 1,000$ kg) launched from the Earth's surface, and reaching an orbit of 10 Earth radii.

3. Escape speed

The gravitational force becomes null only when two objects are at a very large, or infinite, distance. An object's *escape speed* is that which allows an object to escape another object's – often a star or a planet – gravitational pull. Consider as a typical example an object on the surface of the Earth, with a velocity $\boldsymbol{v_i}$. This object is to reach an infinite distance, with a null velocity: conservation of energy requires that

$$E_{K,i} + U_{G,i} = E_{K,f} + U_{G,f} = \tfrac{1}{2}m \cdot v_i^2 - G\tfrac{M_E \cdot m}{R_E} = 0 + 0$$

$$\Rightarrow v_i = \sqrt{\tfrac{2 \cdot G \cdot M_E}{R_E}}$$

which is equal to 10 [km/s]. Things to notice is that the direction of $\boldsymbol{v_i}$ is irrelevant, and that we assumed that no other method of propulsion is possessed by the object, i.e., no forces act on it after it is launched.

Exercise 13.2 Find the escape speed from the Moon, and from planet Jupiter.

4. Kepler's Laws of planetary motion

Many astronomers contributed to the development of our understanding of motion of celestial objects. The motion of planets can be summarized in three laws, which were determined experimentally.

1. Law of orbits: All planets move in elliptical orbits of which the Sun occupies one of the two *foci* [1].

   a: semi-major axis;

   e: *eccentricity* of the orbit. If $e = 0$, the ellipse becomes a circle, and the two foci coincide with the center of the circle. The eccentricity of the Earth's orbit is $e = 0.017$, making the Earth's orbit nearly circular.

2. Law of Areas: The line connecting the planet to the Sun sweeps equal areas in equal time intervals.

   This law can be understood by considering the two areas in figure, which approximately have the same area. For a small time interval, the areas are approximately triangular, and one has

[1] *Foci* is plural for the Latin word *focus*.

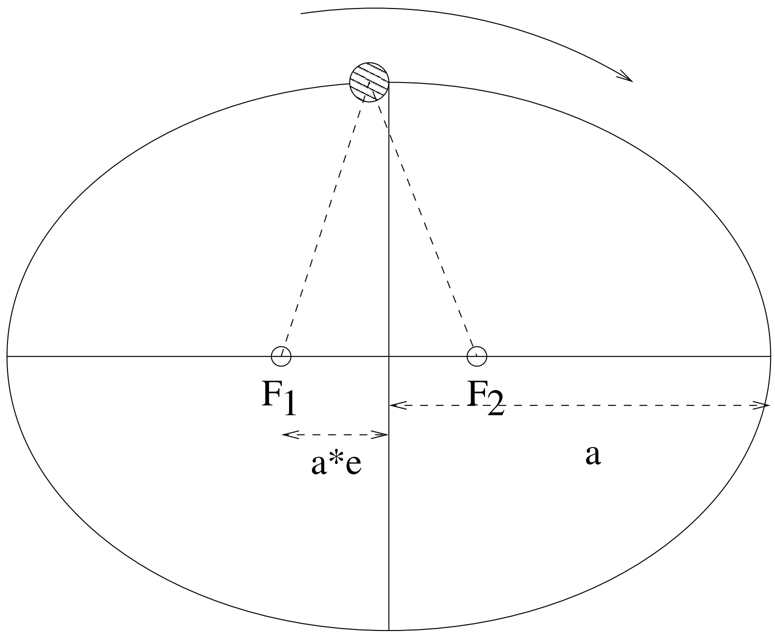

Fig. 4.— An ellipes is a geometrical figure constructed as follows: every point P in the ellipse is such that the *sum of the distances* from P to the two foci, F_1 and F_2, is constant. This sum of distances is represented as the dotted lines.

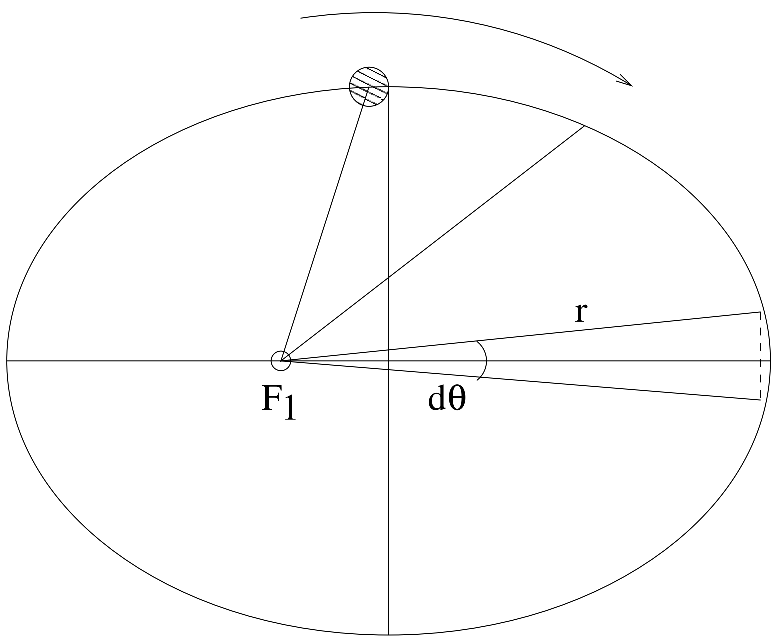

Fig. 5.— Law of areas

$dA \simeq \frac{1}{2}r \cdot (rd\theta)$

where $r \cdot d\theta$ is the short side of the triangle;

$\Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega^2$

where ω is the angular speed of the planet, assuming that the orbit is circular. A more rigorous derivation can be done for elliptical orbits. Since the planet's angular momentum is $L = r \cdot m \cdot v = r^2 \cdot m \cdot \omega$, one obtains that

$\frac{dA}{dt} = \frac{L}{2 \cdot m}$=constant,

because the total angular momentum L is conserved.

The reason by angular momentum should be conserved is the following. The only force acting on the planet is the gravitational force, which is in the direction of the radius. On the other hand, the velocity of the planet is tangent to the orbit, and therefore perpendicular to the radius. The only way to change $\boldsymbol{L}$ is that there is a torque $\boldsymbol{\tau}$, but $\boldsymbol{F_G}$ has null torque with respect to the center of the orbit, as it is directed as $\boldsymbol{r}$.

3. Law of periods: The square of the period T of any planet orbiting the Sun is proportional to the cube of the semi-major axis of that planet's orbit.

   This law is uderstood by using the definition of gravitational force, and recognizing that this force *provides the centripetal force* of a circular orbit:

   $\frac{G \cdot M \cdot m}{r^2} = m \cdot \omega^2 \cdot r$

   $T = \frac{2\pi}{\omega} \Rightarrow \frac{GM}{r^3} = \omega^2$

   $\Rightarrow \frac{T^2}{r^3} = (\frac{4\pi^2}{G \cdot M}$=constant.

   Notice that we approximated the orbit as a circle, in which case the circle equals the semi-major axis of the orbit, a. In the general case of elliptical orbit similar considerations apply, although the derivation is more complex.