CHAPTER 29
Magnetic Fields

1. Origin of magnetic fields

Magnetic fields originate from magnetic materials which, as we have seen, are in turn materials in which electrons rotate in a coherent way. In other words, it seems natural to assume that *currents* are responsible for magnetic fields. This has been proven from an experimental point of view. Consider an element (e.g., of wire) of current i and of length d$\boldsymbol{s}$ along the direction of the current. The Law of *Biót and Savárt* states that such element of current gives rise to the following magnetic field at a distance $\boldsymbol{r}$ from the current element:

$$d\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{i\ d\boldsymbol{s} \times \boldsymbol{r}}{r^3} \tag{1}$$

in which the constant μ_0 is known as the *permittivity* of the medium in which the current and observer are immersed in. In vacuum, this is

$$\mu_0 \simeq 4\pi \cdot 10^{-7}\ \mathrm{T\ m\ A^{-1}} \tag{2}$$

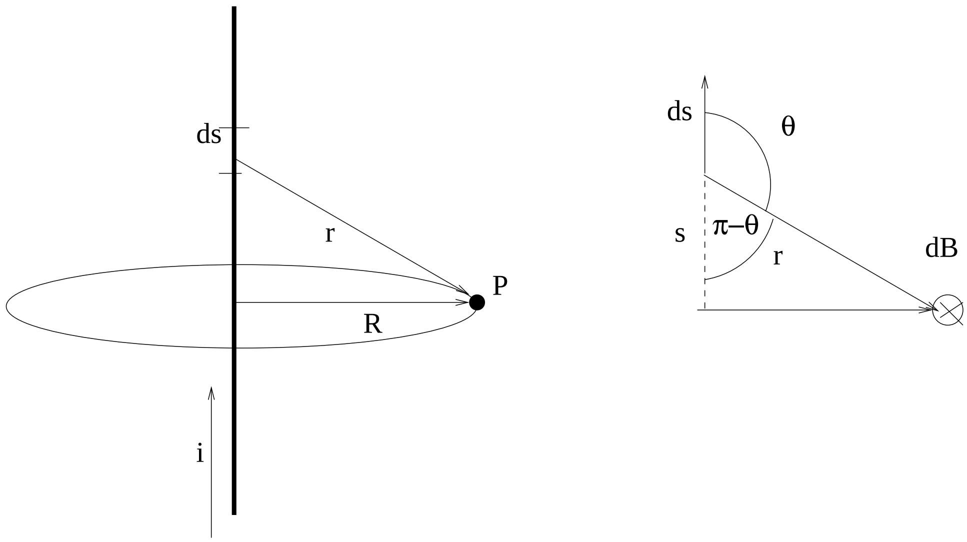
Notice that this law states that the magnetic field approximately drops off with distance as $1/r^2$, similar to the case of the electric field of a point charge and the gravitational field of a point mass. In practice, the calculation of the macroscopic magnetic field $\boldsymbol{B}$ due to a finite distribution of charge can be challenging. In the following we will study a few important cases.

1.1. Magnetic field due to a straight current-carring wire

Consider a vertical wire, as in Figure. Each element of current d$\boldsymbol{s}$ causes a magnetic field

$$dB = \frac{\mu_0}{4\pi}\frac{i\ ds}{r^2} \cdot sin(\theta) \tag{3}$$

If we assume, for the sake of simplicity, that the wire is infinitely long, then we can integrate the previous expression to obtain:



$$B = 2 \cdot \frac{\mu_0 \cdot i}{4\pi} \int_0^{\infty} \frac{sin(\theta) ds}{r^2} \tag{4}$$

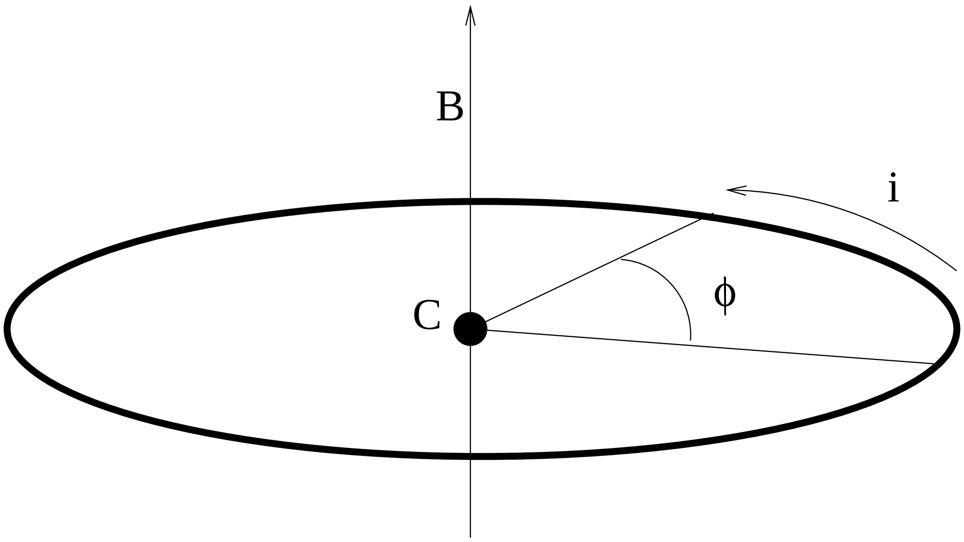
where the integration is on the length ds, and the factor 2 comes from integration both above and below the point of observation P. Given that R is the distance of the point P from the wire, it is true that $r^2 = s^2 + R^2$. Also, $sin(\theta) = sin(\pi - \theta) = R/r = R/\sqrt{s^2 + R^2}$. The integral therefore becomes:

$$B = 2 \cdot \frac{\mu_0 \cdot i}{4\pi} \int \frac{sin(\theta) ds}{r^2} = \frac{\mu_0 \cdot i}{2\pi} \int \frac{R\, ds}{(s^2 + R^2)^{3/2}} = \frac{\mu_0 \cdot i}{2\pi} \left. \frac{s}{(s^2 + R^2)^{1/2}} \right|_0^{\infty} = \frac{\mu_0 \cdot i}{2\pi R} \tag{5}$$

The important feature is the direction of $\boldsymbol{B}$: it is such that it follows the right-hand rule, that is $\boldsymbol{B}$ curls around the current-carrying wire in the counter-clockwise direction. In the case of a *half-infinite wire*, one would perform only one half of the integration above, and one would get a magnetic field of same direction, but magnitude of $B = \mu_0 \cdot i/(4\pi R)$.

1.2. Magnetic field due to an arc or loop

In the case of a current-carrying arc or loop, one may perform a similar integration to that in the previous section. The magnetic field $\boldsymbol{B}$ at the center of the arc or loop loop is as in Figure. The integration (not shown), yields the following result:



$$B = \frac{\mu_0\, i}{4\pi R}\phi \qquad (6)$$

where ϕ is the angle subtended by the arc. If it is a whole loop, then $\phi = 2\pi$, and the equation becomes:

$$B = \frac{\mu_0\, i}{2R} \qquad (7)$$

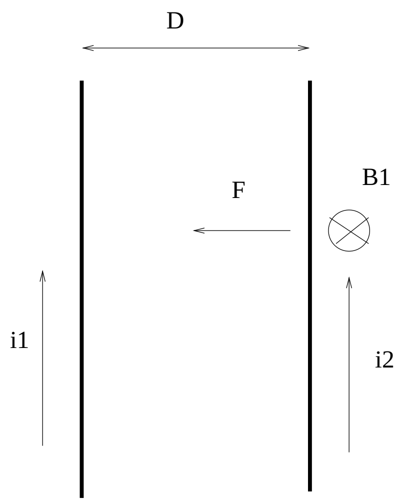
It is important to notice that, once again, the right-hand rule can be used to determine $\boldsymbol{B}$: it is now i that curls around $\boldsymbol{B}$ in the counter-clockwise direction. This shows that the cause (current) and the effect (magnetic field) both obey this simple rule.

2. Forces between current-carrying wires

In the case in which two current-carrying wires are in the vicinity of one another, one can show that there is a mutual attractive/repulsive force experienced by them. Consider, for simplicity, that the two wires are parallel to one another. At the location of wire 2 (to the right), any moving charge will experience a magnetic fielb $\boldsymbol{B_1}$ given by

$$B_1 = \frac{\mu_0 i_1}{2\pi D} \qquad (8)$$

The moving charges on wire 2 will experience a force given by



$$F = i_2 L B_1 \Rightarrow \frac{\mu_0 i_1 i_2 L}{2\pi D} \tag{9}$$

directed as in Figure, where L is the length of the wire. In the case in which both currents i_1 and i_2 are in the same direction, as in this case, the force is attractive: two parallel wires with current in the same direction attract one another! If one of the two currents is in the opposite direction, then the force is repulsive. The other wire also experiences a similar force, again attractive (in figure, to the right).

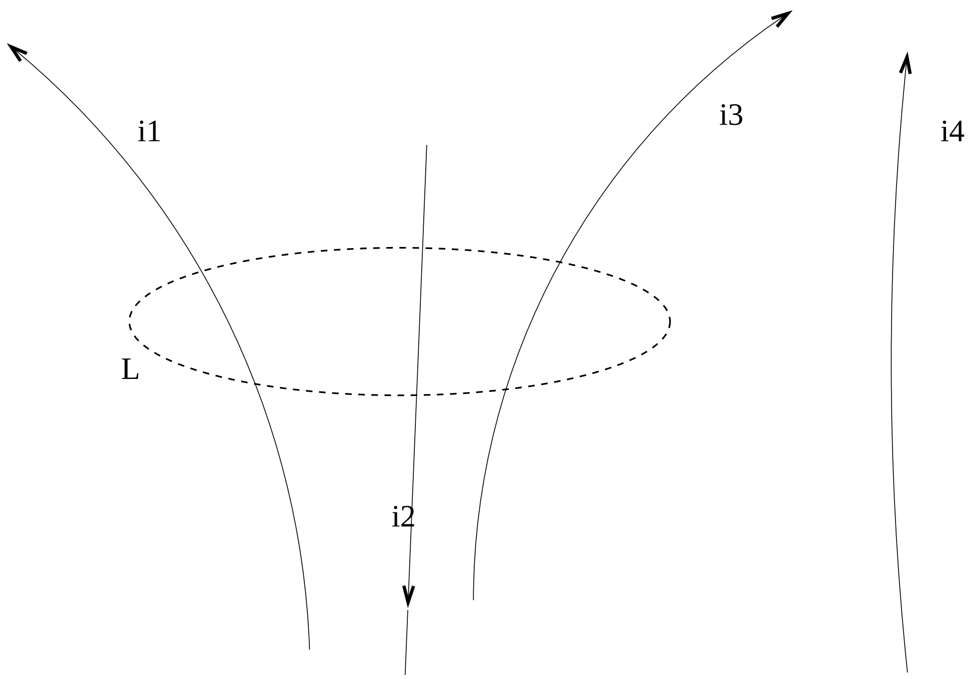
3. Ampere's Law

In calculating the magnetic field due to a complex distribution of currents, one faces a problem similar to the one of calculating an electric field due to a comples charge distribution. This problem can often be alleviated by using Ampere's Law:

$$\oint_L \boldsymbol{B} \cdot d\boldsymbol{s} = \mu_0 \cdot i_{enc} \tag{10}$$

In Equation 10, the simbol $\oint$ indicates a line integral over a closed loop L, called an *Amperian loop*. Th term i_{enc} means the sum of all currents enclosed by the Amperian loop. The method is therefore fully equivalent to that of Gauss' Law:

1) Identify the direction of the magnetic field $\boldsymbol{B}$. This may be often be challenging, but the right-hand rule is useful for this purpose;



2) Identify a suitable Amperian loop, i.e., one which is either perprendicular or parallel to $\boldsymbol{B}$, so that the integration of $\oint_L \boldsymbol{B} \cdot d\boldsymbol{s}$ is straightforward;

3) Count the currents enclosed by the loop L. Since one may chose either direction for the line integral (clockwise or counter-clockwise), I suggest one chooses the counter-clockwise direction, so that currents will be counted as positive if in the direction of the right-hand thumb, and negative otherwise. In the figure above, i_2 would then be negative.

3.1. A straight current-carrying wire

We can test Ampere's Law using a geometry that we have already stuied earlier in section 1.1. In that case, one performs the following steps in order to calculate $\boldsymbol{B}$ at a distance R from the wire (see Figure in section 1.1):

1) Using the right-hand rule, one can anticipate that $\boldsymbol{B}$ will be curling around i in the counter-clockwise direction.

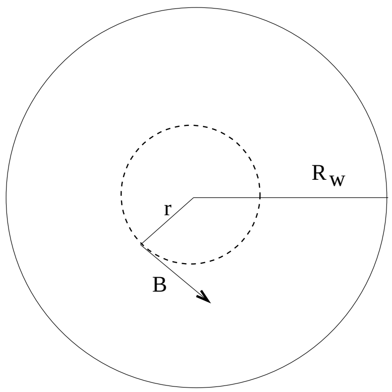
2) Use a circle of radius R, concentric with i, as the Amperian loop:

3) $\oint_L \boldsymbol{B} \cdot d\boldsymbol{s} = B \cdot 2\pi R = \mu_0 \cdot i$

$$\Rightarrow B(R) = \frac{\mu_0 i}{2\pi R} \tag{11}$$

which is exactly what we had found using brute-force integration.

One can extend the study of a current-carrying wire by considering the field $\boldsymbol{B}$ *inside* the wire; assume that the radius of the wire is R_w, let's consider $r < R_w$.



One anticipates again the direction of $\boldsymbol{B}$, and a simple integration over the Amperial loop - a circle of radius r - gives:

$\oint_L \boldsymbol{B} \cdot d\boldsymbol{s} = B \cdot 2\pi r = \mu_0 \cdot i(r)$

The key is understanding what $i(r)$ means: it is the charge *within* the radius r of the wire. If one assumes that the wire carries a uniform current throughout, then it is clear that the number of charges dQ crossing a given point in time dt is proportional to the area of the wire; therefore

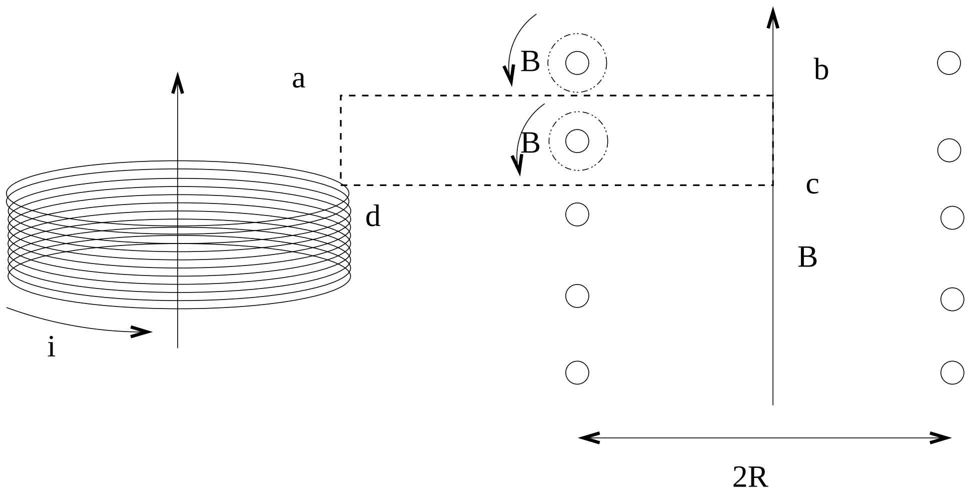
$$i(r) = i \cdot \frac{\pi r^2}{\pi R_w^2} = i \cdot \frac{r^2}{R_w^2} \tag{12}$$

$$\Rightarrow B(r) \cdot 2\pi r = \mu_0 i \cdot \frac{r^2}{R_w^2} \Rightarrow B(r) = \mu_0 i \cdot \frac{r}{2\pi R_w^2} \tag{13}$$

Notice that the value of the permeability constant μ_0 is that of the material that consitutes the wire, e.g., copper.

3.2. Magnetic field in a solenoid

A solenoid is a continuous wire composed of several tightly wound loops, where a current i flows. Inside the solenoid, according to the right-hand rule, one expects the magnetic field



to be vertical, i.e., axial (along the axis of the solenoid). Outside of the solenoid, one expects to have a magnetic field, with axial direction; in fact, the field can be though of as the combination of the fields of each current element.

The choice of the Amperian loop is somewhat complicated by the presence of the field outside of the solenoid. However, if one chooses a square *abcda* as in Figure, with the side *ad* at a large distance from the solenoid (not to scale in the figure), then one can use Ampere's Law as follows:

$$\oint_L \boldsymbol{B} \cdot d\boldsymbol{s} = \int_{ab} \boldsymbol{B} \cdot d\boldsymbol{s} + \int_{bc} \boldsymbol{B} \cdot d\boldsymbol{s} + \int_{cd} \boldsymbol{B} \cdot d\boldsymbol{s} + \int_{da} \boldsymbol{B} \cdot d\boldsymbol{s} \qquad (14)$$

One can solve each integral:

$\int_{ab} \boldsymbol{B} \cdot d\boldsymbol{s}$=0, since $\boldsymbol{B}$ is perpendicular to $d\boldsymbol{s}$;

$\int_{bc} \boldsymbol{B} \cdot d\boldsymbol{s} = B \cdot h$, since $\boldsymbol{B}$ is parallel to $d\boldsymbol{s}$, and h is the length of segment bc;

$\int_{cd} \boldsymbol{B} \cdot d\boldsymbol{s}$=0, since $\boldsymbol{B}$ is perpendicular to $d\boldsymbol{s}$;

$\int_{da} \boldsymbol{B} \cdot d\boldsymbol{s}$=0, since $\boldsymbol{B}$ is small in magnitude (although parallel to $d\boldsymbol{s}$);

$$\oint_L \boldsymbol{B} \cdot d\boldsymbol{s} = \mu_0 i_{enc} \qquad (15)$$

in which $i_{enc} = i \cdot (nh)$, where n is the number of turns per unit length, and i is the current through the solenoid.

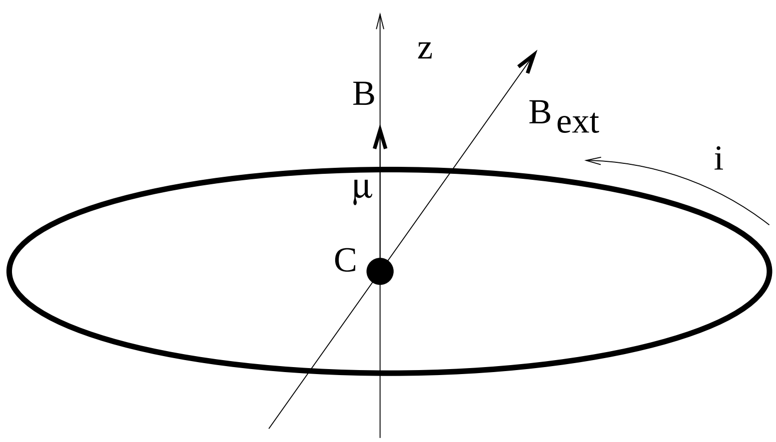
$$B = \mu_0 \cdot ni \tag{16}$$

The result means that the larger the number of turns per unit lenght, the larger the magnetic field, which is a sensible result.

3.3. A current-carrying loop, or coil

This geometry was already treated earlier, finding that the magnetic field at the center of a coil of radius R (or current-carrying loop) is given by

$$B = \frac{\mu_0 \, i}{2R} \tag{17}$$



Along the axis of the loop , but at a large distance from the plane of the loop, one can show that the magnetic field is given by

$$B(z) = \frac{\mu_0 \, i \, R^2}{2(R^2 + z^2)^{3/2}} \simeq \frac{\mu_0 i R^2}{2z^3} \Rightarrow \boldsymbol{B(z)} = \frac{\mu_0}{2\pi} \frac{\boldsymbol{\mu}}{z^3} \tag{18}$$

where $\boldsymbol{\mu}$ is the magnetic dipole moment, not to be confused with the permeability constant μ. Te proof of this equation can be provided by direct integration of the Biot and Savart Law. Moreover, one can show that there is a magnetic field all around the coil (see, e.g., Fig. 29-22 of textbook).

The importance of the coil is in the following. Firts, the coil generates a magnetic field $\boldsymbol{B}$, given by Equation 18 for points along its axis. Also, if there is an external field $\boldsymbol{B_{ext}}$ present, generated by other means, then the coil experiences a torque given by

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B_{ext}} \tag{19}$$

Given the direction of $\boldsymbol{\mu}$, the torque $\boldsymbol{\tau}$ is such that the coil tends to align $\boldsymbol{\mu}$ with $\boldsymbol{B_{ext}}$, and therefore $\boldsymbol{B_{ext}}$ with $\boldsymbol{B}$. This shows the fact that the coil's $\boldsymbol{B}$ tends to align itself with the external magnetic field $\boldsymbol{B_{ext}}$.