There are four problems for 50 points. Please show your work in a well organized way. No work, no credit.

1. (16 points) Mark each statement True or False. Justify each answer.

   (1). We can always define an order relation on a field to make it an ordered field.
       **False.** For example, we cannot introduce an order relation to make \( \mathbb{C} \), the set of complex numbers, an ordered field. This is based on Problem 11.10.

   (2). Every nonempty finite set has a maximum and a minimum.
       **True.** Note that a nonempty finite set contains only finitely many elements (remark: \((0, 1)\) is not a finite set!), by sorting all the elements in the set, we know that it has a maximum and a minimum.

   (3). Suppose \( S \subseteq \mathbb{R} \) contains infinitely many points, then \( S' \neq \emptyset \).
       **False.** For example, \( \mathbb{N} \) contains infinitely many points, but \( \mathbb{N}' = \emptyset \).

   (4) Let \( I \) be the set of isolated points of \( S \subseteq \mathbb{R} \), then \( \text{int} S \cap I = \emptyset \).
       **True.** Because \( \text{int} S \subseteq S' \) and \( S' \cap I = \emptyset \).

2. (12 points)

   (1). Prove the sum of a rational number and an irrational number is irrational.

   (2). Prove that for any two distinct real numbers \( x < y \), there exists an irrational number of the form \( r + \sqrt{3} \) sitting between \( x \) and \( y \), where \( r \in \mathbb{Q} \) is rational.

   **Proof**

   (1). Let \( r \in \mathbb{Q} \) and \( w \in \mathbb{R} \setminus \mathbb{Q} \), we prove \( r + w \in \mathbb{R} \setminus \mathbb{Q} \) by contradiction. Suppose \( r + w \in \mathbb{Q} \), we have \( w = (w + r) - r \in \mathbb{Q} \) since \( \mathbb{Q} \) is a field. That contradicts that \( w \in \mathbb{R} \setminus \mathbb{Q} \). Therefore, \( r + w \) is irrational.

   (2). Since \( x < y \), we have \( x - \sqrt{3} < y - \sqrt{3} \). By the Density of Rational Numbers, \( \exists r \in \mathbb{Q} \) such that \( x - \sqrt{3} < r < y - \sqrt{3} \), which implies \( x < r + \sqrt{3} < y \).  \( \Box \)
3. (12 points) Let \( A \subseteq \mathbb{R} \) and \( B \subseteq \mathbb{R} \) be two bounded subsets of \( \mathbb{R} \). Define
\[
C = \{ x - y : x \in A, y \in B \}
\]
Prove that \( C \) is bounded in \( \mathbb{R} \), and represent \( \sup C \) in terms of the bounds of \( A \) and \( B \).

**Proof** Since \( A \subseteq \mathbb{R} \) and \( B \subseteq \mathbb{R} \) are bounded, by the Completeness Axiom of \( \mathbb{R} \), we have \( \inf A, \sup A, \inf B \) and \( \sup B \) exist and are real. Furthermore, \( \forall x \in A, \forall y \in B \), we have \( \inf A \leq x \leq \sup A \), \( \inf B \leq y \leq \sup B \).

By the way we define \( C \), \( \forall c \in C \), \( \exists x_c \in A \) and \( y_c \in B \) such that \( c = x_c - y_c \), which gives us
\[
\inf A - \sup B \leq c \leq \sup A - \inf B.
\]

By the definition of a bounded set and the completeness, we prove \( C \) is bounded, and
\[
\inf A - \sup B \leq c \leq \sup A - \inf B.
\]

Note that \( \forall \varepsilon > 0, \exists x_\varepsilon \in A \) such that \( x_\varepsilon > \sup A - \varepsilon \), and for the above \( \varepsilon \), \( \exists y_\varepsilon \in B \) such that \( y_\varepsilon < \inf B + \varepsilon \). Therefore, noting that \( x_\varepsilon - y_\varepsilon \in C \), we have
\[
\sup C \geq x_\varepsilon - y_\varepsilon \geq (\sup A - \varepsilon) - (\inf B + \varepsilon) = \sup A - \inf B - 2\varepsilon,
\]
which implies
\[
\sup C \geq \sup A - \inf B.
\]
Consequently, we have
\[
\sup C = \sup A - \inf B.
\]
\[\Box\]

**Remark:** To make the answer shorter, a key observation is that \( C \) is the Minkowski sum of \( A \) and \( -B \). By Theorem 12.7 and Problem 12.7, we have
\[
\sup C = \sup A + \sup(-B) = \sup A - \inf B.
\]

4. (10 points) Let \( S \subseteq \mathbb{R} \). Prove that if \( x \in S' \cap (\mathbb{R}\setminus S) \), then \( x \in \text{bd}S \).

**Proof** If \( x \in S' \cap (\mathbb{R}\setminus S), \forall \varepsilon > 0 \), by \( x \in S \), we have \( N(x, \varepsilon) \cap S \supseteq N^*(x, \varepsilon) \cap S \neq \emptyset \); by \( x \in \mathbb{R}\setminus S \), we have \( N(x, \varepsilon) \cap (\mathbb{R}\setminus S) \supseteq \{x\} \neq \emptyset \). Therefore, \( x \in \text{bd}S \). \[\Box\]