MA 452/502: Introduction to Real Analysis
Exam 1: Solutions

July 2, 2007

1. (40 points) Mark each statement True or False. Justify each answer.

(1). If x, y, $z \in \mathbb{R}$ and $x < y$, then $xz < yz$.

False. Example: Let $x = 1$, $y = 2$, $z = -1$, then $x < y$ but $xz > yz$.

(2). Every nonempty bounded subset of $\mathbb{R}$ has a maximum and a minimum.

False. Example: Let $S = (0, 1)$, which is nonempty and bounded, but it has no maximum (and minimum).

(3). Every deleted neighborhood of $x \in \mathbb{R}$ is an open set.

True. Since any deleted neighborhood of x is a union of two open intervals.

(4). No infinite set is compact.

False. Example: Let $S = [0, 1]$, which is an infinite set, and compact by Heine-Borel Theorem.

(5). If a set has a maximum and a minimum, then it is compact.

False. Example: Let $S = [0, 1) \cup (2, 3]$, which has a maximum and a minimum but it is not compact.

(6). Let $S \subseteq \mathbb{R}$, then $\text{int}S \cap \text{bd}S = \emptyset$.

True. For $x \in \text{int}S$, which means $N(x, \varepsilon_0) \subseteq S$ for some $\varepsilon_0 > 0$, we have $N(x, \varepsilon_0) \cap \big(\mathbb{R}\backslash S\big) = \emptyset$ and $x \notin \text{bd}S$ by Definition 13.3. Therefore, $\text{int}S \cap \text{bd}S = \emptyset$.

(7). If for every $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that for every $n > N$ we have $s_n < \varepsilon$, then $\lim_{n\to\infty} s_n = 0$.

False. Example: Let $s_n = -1$ for all $n \in \mathbb{N}$, then for every $\varepsilon > 0$ there exists $N = 0 \in \mathbb{R}$ such that for every $n > N$ we have $s_n < \varepsilon$ but $\lim_{n\to\infty} s_n = -1 \neq 0$.

(8). If (s_n) and (t_n) converge to s and t, respectively, and $s_n < t_n$ for every $n \in \mathbb{N}$, then $s < t$.

False. Example: Let $s_n = 0$ and $t_n = \frac{1}{n}$ for $n \in \mathbb{N}$, then $s_n < t_n$ but $s = t = 0$.

(9). A sequence is bounded and monotone if and only if it is a Cauchy sequence.

False. Example: Let $s_n = \frac{(-1)^n}{n}$ for $n \in \mathbb{N}$, which is a Cauchy sequence (since it converges to 0) but it is not monotone.

(10). Every bounded sequence has a convergent subsequence.

True. By Theorem 19.7.

2. (10 points) Let S be a bounded infinite set and let $s = \sup S$. Prove: If $s \notin S$, then $s \in S'$.

Proof By the definition of s, $\forall \varepsilon > 0\ \exists a \in S$ such that $s \geq a > s - \varepsilon$, which implies $a \in N(s, \varepsilon)$. Since $s \notin S$, we have $a \neq s$ and $a \in N^*(s, \varepsilon) \cap S$. By the definition of S', we have $s \in S'$. □

3. (14 points)

(1). Prove the following statement by using only the definition:

$$\lim_{n\to\infty} \frac{\sin(\pi^2 n)}{n} = 0.$$

Proof Since $\forall \varepsilon > 0$, $\exists N = \frac{1}{\varepsilon} \in \mathbb{R}$, such that $\forall n > N$, we have

$$\left| \frac{\sin(\pi^2 n)}{n} - 0 \right| \leq \frac{1}{n} < \varepsilon.$$

By Definition 16.2, we prove our conclusion. □

(2). Prove: The sequence (s_n) defined by $s_n = \cos(n\pi)$ is divergent.

Proof By choosing $n = 2k$, we have $s_{2k} = 1 \to 1$ as $k \to \infty$, and by choosing $n = 2k-1$, we have $s_{2k-1} = -1 \to -1$ as $k \to \infty$. Therefore, by Theorem 19.4 (truly, Example 19.6), we prove that (s_n) defined by $s_n = \cos(n\pi)$ is divergent. (Of course, we can prove the statement based on Example 16.12 as well.) □

4. (18 points) Find the following limits.

(1). $\lim_{n\to\infty} \left(\sqrt{n^2+2n} - \sqrt{n^2-3n}\right)$

Solution.

$$\begin{aligned}
&\lim_{n\to\infty} \left(\sqrt{n^2+2n} - \sqrt{n^2-3n}\right) \\
&= \lim_{n\to\infty} \frac{\left(\sqrt{n^2+2n} - \sqrt{n^2-3n}\right)\left(\sqrt{n^2+2n} + \sqrt{n^2-3n}\right)}{\sqrt{n^2+2n} + \sqrt{n^2-3n}} \\
&= \lim_{n\to\infty} \frac{5n}{\sqrt{n^2+2n} + \sqrt{n^2-3n}} \\
&= \frac{5}{2}.
\end{aligned}$$

(2). $\lim_{n\to\infty} \frac{2n^2-3}{1-2n}$

Solution. Since the degree of the numerator is bigger than the degree of the denominator, and the product of the leading coefficients is negative, we have

$$\lim_{n\to\infty} \frac{2n^2-3}{1-2n} = -\infty.$$

(3). $\lim_{n\to\infty} \frac{n^2}{1.01^n}$

Solution. Since

$$\frac{s_{n+1}}{s_n} = \frac{\frac{(n+1)^2}{1.01^{n+1}}}{\frac{n^2}{1.01^n}} = \frac{(n+1)^2}{1.01n^2} \to \frac{1}{1.01} < 1 \quad \text{as } n \to \infty,$$

by Theorem 17.7, we have $\lim_{n\to\infty} \frac{n^2}{1.01^n} = 0$.

5. (18 points) Let (s_n) be defined by $s_1 = \sqrt{2}$, $s_{n+1} = \sqrt{2+s_n}$ for $n \in \mathbb{N}$.

(1). Prove that (s_n) is an irrational sequence, that is, for every $n \in \mathbb{N}$, s_n is an irrational number.

Proof We will prove the statement by induction. Since $s_1 = \sqrt{2}$ is irrational (by Theorem 12.1), the statement is true for $n = 1$. Suppose the statement is true for $n = k$, that is, s_k is irrational, we will prove that s_{k+1} is irrational. If not, suppose s_{k+1} is rational, by $s_{k+1} = \sqrt{2+s_k}$, we have $s_k = s_{k+1}^2 - 2$ is rational (Why?), which contradicts that s_k is irrational. Therefore, s_{k+1} is irrational. By induction, we prove the result. □

(2). Prove (s_n) is monotone and bounded.

Proof This is almost the same as the proof of Example 18.4, therefore omitted here. □

(3). Find $\lim_{n\to\infty} s_n$.

Solution. By (2), we know $\lim s_n = s \in \mathbb{R}$. From $s_{n+1} = \sqrt{2+s_n}$, we have $s = \sqrt{2+s}$. Solving the radical equation, we get $s = 2$ or $s = -1$. Since s_n is increasing with $s_1 = \sqrt{2}$, we have $s = 2$.