MA 585: Probability
Solutions to Exam 1

1. (10 points) Consider a coin-die experiment: One flips a fair coin at first. If he gets a head, then he will roll a 6-sided fair die; otherwise, he will roll a 6-sided unfair die, which has probability $\frac{7-i}{21}$ to get i faces up ($i = 1, \ldots, 6$). If one gets a 2 faces up, what is the probability that he got a tail when he flipped the coin?

Solution: Let $A = \{2 \text{ faces up}\}$, $H = \{\text{Head}\}$ and $T = \{\text{Tail}\}$. Then we have $\mathbb{P}\{H\} = \mathbb{P}\{T\} = 1/2$, $\mathbb{P}\{A|T\} = 5/21$ and $\mathbb{P}\{A|H\} = 1/6$. Therefore,

$$\mathbb{P}\{T|A\} = \frac{\mathbb{P}\{A|T\}\mathbb{P}\{T\}}{\mathbb{P}\{A|T\}\mathbb{P}\{T\} + \mathbb{P}\{A|H\}\mathbb{P}\{H\}} = \frac{10}{17}.$$

2. (10 points) Suppose the test score that a student gets is uniformly distributed over [40, 100]. What is the probability that no more than one student failed the test among a group of 10 students who took the test? (We think one gets no less than 60 as a pass.)

Solution: The probability that a student passes the test is

$$p = \int_{60}^{100} \frac{1}{60} dx = \frac{2}{3}.$$

Let X be the number of students from a group of 10 students who pass the test, then $X \sim \text{Bin}(10, 2/3)$, and

$$\mathbb{P}\{\text{no more than one failed}\} = \mathbb{P}\{X \geq 9\} = \left(\frac{2}{3}\right)^{10} + 10\left(\frac{2}{3}\right)^{9}\frac{1}{3} = \frac{2^{11}}{3^9}.$$

3. (10 points) A series of Bernoulli trials with successful rate $p \in (0, 1)$ is performed. We will stop the experiment whenever a changeover occurs, which means that the outcome differs from the one preceding it. Let X denote the number of Bernoulli trials being performed.

(1). Prove that $\mathbb{P}\{X \geq 3\} \geq 0.5$.

(2). Find $\mathbb{E}[X]$.

Solution:

(1). Prove that $\mathbb{P}\{X \geq 3\} \geq 0.5$.

The possible values of X are 2, 3, ..., and $\mathbb{P}\{X = 2\} = p(1-p)+(1-p)p = 2p(1-p) \leq 1/2$. Therefore,

$$\mathbb{P}\{X \geq 3\} = 1 - \mathbb{P}\{X = 2\} \geq 1 - 1/2 = 0.5.$$

(2). Find $\mathbb{E}[X]$.

Notice that

$$\mathbb{P}\{X = k\} = p^{k-1}(1-p) + (1-p)^{k-1}p, \quad k = 2, 3, \ldots,$$

we have

$$\mathbb{E}[X] = \sum_{k=2}^{\infty} k\mathbb{P}\{X = k\} = \frac{2p - p^2}{1-p} + \frac{2(1-p) - (1-p)^2}{p} = \frac{1}{p(1-p)} - 1.$$

4. (10 points) Let the probability density function of X be defined by $f(x; \theta, \lambda) = ce^{-|x-\theta|\lambda}$, where $\lambda > 0$, $\theta \in \mathbb{R}$ are the scale parameter and location parameter, respectively. Find c and $\mathbb{P}\{|X - 1| < 2\}$.

Solution: From

$$\int_{-\infty}^{\infty} f(x)dx = 1,$$

we can determine that $c = \lambda/2$. Now

$$\begin{aligned} \mathbb{P}\{|X-1| < 2\} &= \mathbb{P}\{-1 < X < 3\} \\ &= \begin{cases} \frac{e^{(1+\theta)\lambda} - e^{(\theta-3)\lambda}}{2} & \theta < -1 \\ 1 - \frac{1}{2}\left[e^{-(1+\theta)\lambda} + e^{(\theta-3)\lambda}\right] & -1 \leq \theta < 3 \\ \frac{e^{(3-\theta)\lambda} - e^{-(1+\theta)\lambda}}{2} & \theta \geq 3 \end{cases} \end{aligned}$$

5. (10 points) Let X be a nonnegative continuous random variable, prove that for any $t > 0$,

$$\mathbb{P}\{X > t\} \leq \frac{\mathbb{E}[X]}{t}.$$

Solution:

$$\begin{aligned} \mathbb{P}\{X > t\} &= \int_t^{\infty} f(x)dx \\ &\leq \int_t^{\infty} \frac{x}{t} f(x)dx \\ &= \frac{1}{t}\int_t^{\infty} xf(x)dx \\ &\leq \frac{1}{t}\int_0^{\infty} xf(x)dx = \frac{\mathbb{E}[X]}{t}. \end{aligned}$$

6. (10 points) Suppose the number of customers who enter a post office over [0, 1] has a Poisson(λ) distribution. Let X be the time that the first customer enters the post office. Find

$\mathbb{P}\{X \leq 1\}$.
Solution: Let N be the number of customers who enter a post office over [0, 1], then N ∼Poisson(λ). Notice that $\{X > 1\}$=$\{N = 0\}$, which means that the first customer enters the post office after 1 iff the number of customers entered the office within [0, 1] is 0, we have

$$\mathbb{P}\{X \leq 1\} = 1 - \mathbb{P}\{X > 1\} = 1 - \mathbb{P}\{N = 0\} = 1 - e^{-\lambda}.$$