MA 452/502: Introduction to Real Analysis
Solutions to Exam 2

October 18, 2007

There are five problems for 50 points. Please show your work in a well organized way. No work, no credit.

1. (16 points) Mark each statement True or False. Justify each answer.

(1). A sequence $\big(s_n\big)$ is convergent iff $\big(|s_n|\big)$ converges.

False. Consider $s_n = (-1)^n$. Clearly $\big(|s_n|\big) = \big(1\big)$ which converges, but $\big(s_n\big)$ itself is divergent.

(2). If $s_n \to s$, $t_n \to t$, and $s_n < t_n$ for all $n \in \mathbb{N}$, then $s < t$.

False. Let $s_n = \frac{1}{n}$ and $t_n = \frac{2}{n}$, then $s_n \to 0$, $t_n \to 0$ and $s_n < t_n$ for all $n \in \mathbb{N}$ but $s = t = 0$.

(3). A monotone sequence $\big(s_n\big)$ is Cauchy iff it is bounded.

True. This is because of the Monotone Convergence Theorem (Theorem 18.3) and the Cauchy Convergence Criterion (Theorem 18.12).

(4) Let $\big(s_n\big)$ be a sequence. If the set $S := \{s_n,\ n \in \mathbb{N}\}$ is finite, then $\big(s_n\big)$ has a convergent subsequence.

True. Since S is finite, it is bounded. The conclusion follows from Theorem 19.7.

2. (8 points) Let $\alpha > 1$. Prove the following statement by using only the definition of limit:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \sin k}{n^\alpha} = 0.$$

Proof Since $\forall \varepsilon > 0$, $\exists N(\varepsilon) = \varepsilon^{\frac{1}{1-\alpha}}$ such that $\forall n > N$, we have

$$\left| \frac{\sum_{k=1}^{n} \sin k}{n^\alpha} \right| \leq \frac{\sum_{k=1}^{n} |\sin k|}{n^\alpha} \leq \frac{n}{n^\alpha} < \varepsilon,$$

where we have used the facts that $|\sin k| \leq 1$ for all $k \in \mathbb{N}$ and $\alpha > 1$.

By definition, we prove that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \sin k}{n^\alpha} = 0.$$

□

3. (10 points) Find the following limits

(1). $\lim_{n\to\infty} \frac{n!}{n^2}$ (2). $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+3n}-n}$

Solution.

(1). Since $s_n := \frac{n!}{n^2} > 0$ and

$$\frac{s_n}{s_{n+1}} = \frac{n!(n+1)^2}{(n+1)!n^2} = \frac{(n+1)}{n^2} \to 0 < 1,$$

we have, by Theorem 17.7,

$$\lim_{n\to\infty} \frac{1}{s_n} = 0.$$

By Theorem 17.13, we have $\lim_{n\to\infty} \frac{n!}{n^2} = +\infty$.

(2).

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+3n}-n} = \lim_{n\to\infty} \frac{\sqrt{n^2+3n}+n}{3n} = \frac{2}{3}.$$

4. (8 points) Let $s_n = \cos\left(\frac{n\pi}{2}\right)$. Prove that the sequence $\left(s_n\right)$ is divergent.

Proof Since $\left(s_{2k+1}\right) = (0)$, we have $\lim_{k\to\infty} s_{2k+1} = 0$. Since $\left(s_{4k}\right) = (1)$, we have $\lim_{k\to\infty} s_{4k} = 1$. Therefore, we prove $\left(s_n\right)$ is divergent by Theorem 19.4. □

5. (8 points) Let

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Prove that the sequence $\left(s_n\right)$ is convergent.

Remark. We may prove the result by proving that $\left(s_n\right)$ is Cauchy, or by noting that it is increasing and proving that $\left(s_n\right)$ is bounded.

Proof (Cauchy Criterion) We will prove $\left(s_n\right)$ is convergent by proving that it is Cauchy. Since $\forall \varepsilon > 0$, $\exists N(\varepsilon) = \varepsilon^{-1}$ such that $\forall n$, $m > N$ (here we may assume $m > n$), we have

$$|s_n - s_m| = \sum_{k=n+1}^{m} \frac{1}{k^2} \leq \sum_{k=n+1}^{m} \frac{1}{(k-1)k} = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \varepsilon,$$

which proves $\left(s_n\right)$ is Cauchy, and the conclusion follows from Theorem 18.12. □

Proof (Boundedness) Note that $\left(s_n\right)$ is increasing, we only need to prove $\left(s_n\right)$ is bounded above. In the following, we prove $s_n < 2$ for all $n \in \mathbb{N}$. In fact

$$s_n = 1 + \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \frac{1}{(k-1)k} \leq 1 + 1 - \frac{1}{n} < 2.$$

□