MA 452/502: Introduction to Real Analysis
Solutions to Exam 2

July 30, 2007

There are five problems in this exam for a maximum of 100 points. Please show your work and present your solutions in a well organized way. No work, no credit.

1. (40 points) Mark each statement True or False. Justify each answer.

(1). Let $f : D \to \mathbb{R}$ and $c \in D' \cap D$, then f has a limit at c iff f is continuous at c.

False. Let $f(x) = x$ for $x \neq 0$ and $f(0) = 1$, then $f(x)$ has a limit 0 at 0, but it is not continuous at 0.

(2). Let $f : \mathbb{N} \to \mathbb{R}$, then f is continuous on $\mathbb{N}$.

True. That is because all the points in $\mathbb{N}$ are isolated points of $\mathbb{N}$.

(3). Let $f : D \to \mathbb{R}$ and define $|f| : D \to \mathbb{R}$ by $|f|(x) = |f(x)|$, and $f^2 : D \to \mathbb{R}$ by $f^2(x) = \big(f(x)\big)^2$. If $|f|$ is continuous at $c \in D$, then $f^2(x)$ is continuous at c.

True. That is because of the product rule of continuous functions. Note that $f^2(x) = |f|(x) \cdot |f|(x)$, and $|f|(x)$ is continuous at c, we have $f^2(x)$ is continuous at c.

(4). Every continuous function on a bounded interval is bounded.

False. Let $f(x) = \frac{1}{x}$ for $x \in (0,\, 1)$, which is a continuous function on a bounded interval, but it is not bounded.

(5). Equation $x^{2007} + 1 = 4x^{1006}$ has at least one real solution in $(0,\, 1)$.

True. Define $f(x) = x^{2007} - 4x^{1006} + 1$, which is a polynomial function. Since $f(0) = 1 > 0$, $f(1) = -2 < 0$, by the Intermediate Value Theorem, $\exists c \in (0,\, 1)$ such that $f(c) = 0$, that is, $x^{2007} + 1 = 4x^{1006}$ has at least one real solution in $(0,\, 1)$.

(6). Let c be a point in the interval I, and suppose $f : I \to \mathbb{R}$. If $f'(c) > 0$, then f is continuous at c.

True. That is due to Theorem 25.6.

(7). Let c be a point in the interval I, and suppose $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$. If f and g are differentiable at c, then the composite function $g \circ f$ is differentiable at c.

False. Let $f(x) = -x$, $g(x) = \sqrt{x}$ for $x \in (0,\, 1)$, then f and g are differentiable on $(0,\, 1)$, but $g \circ f$ is undefined on $(0,\, 1)$.

(8). Suppose f and g are continuous on $[0,\,10]$, and differentiable on $(0,\,10)$. If $f'(x) = g'(x)$ for all $x \in (0,\,10)$ and $f(\pi) = g(\pi)$, then $f(x) = g(x)$.

True. Let $h(x) = f(x) - g(x)$, then h is continuous on $[0,\,10]$ and differentiable on $(0,\,10)$, and $h'(x) = 0$ for all $x \in (0,\,10)$. By Theorem 26.6, $h(x) = Constant$. Since $h(\pi) = 0 = Constant$, we have $h(x) = 0$, and therefore $f(x) = g(x)$.

(9). Let $f : [a,\,b] \to \mathbb{R}$ be bounded and define $|f| : [a,\,b] \to \mathbb{R}$ by $|f|(x) = |f(x)|$. If $|f|$ is integrable on $[a,\,b]$, then f is integrable on $[a,\,b]$.

False. Let $f(x) = 1$ if $x \in \mathbb{Q} \cap [0,\,1]$ and $f(x) = -1$ if $x \in [0,\,1] \backslash \mathbb{Q}$. Then f is not integrable on $[0,\,1]$. But $|f|(x) = 1$ for all $x \in [0,\,1]$, which is integrable.

(10). If f is neither monotone nor continuous on $[a,\,b]$, then f is not integrable on $[a,\,b]$.

False. Let $f(x) = 2 - x$ for $x \in [0,\,1]$ and $f(x) = 4x + 3$ for $x \in (1,\,2]$, then f is integrable on $[0,\,2]$ but neither monotone nor continuous on $[0,\,2]$

2. (24 points)

(1). Prove the following statement by using only the definition:

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Proof Clearly $f(x) := x \sin\left(\frac{1}{x}\right)$ is well defined for all $x \neq 0$. For any $\varepsilon > 0$, $\exists \delta = \varepsilon$ such that whenever $0 < |x| < \delta$, we have

$$\left| x \sin\left(\frac{1}{x}\right) - 0 \right| \leq |x| < \varepsilon.$$

By Definition 20.1,

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

□

(2). Let

$$f(x) = \begin{cases} 2x + 5 & \text{if } x \leq 1, \\ 9x^2 - 2 & \text{if } x > 1. \end{cases}$$

Prove that $f(x)$ is continuous but NOT differentiable at 1.

Proof Clearly, $f(x)$ is defined on $\mathbb{R}$. Since

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \big(2x + 5\big) = 7, \;\; \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \big(9x^2 - 2\big) = 7,$$

we have $\lim_{x \to 1} f(x) = f(1) = 7$. By Theorem 21.2, $f(x)$ is continuous at 1.

To see that $f(x)$ is not differentiable at 1, we compute the left and right derivatives of $f(x)$ at 1. Since

$$f'_{-}(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{2x-2}{x-1} = 2,\ f'_{+}(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{9x^2-9}{x-1} = 18,$$

we have $f'_{-}(1) \neq f'_{+}(1)$, which implies f is not differentiable at 1. ☐

(3). Let $f(x) = \sqrt{2x-1}$, find $f'(2)$ by using only the definition.

Solution

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2} \frac{\sqrt{2x-1}-\sqrt{3}}{x-2} \\ &= \lim_{x\to 2} \frac{2(x-2)}{(x-2)(\sqrt{2x-1}+\sqrt{3})} = \frac{1}{\sqrt{3}}. \end{aligned}$$

3. (12 points) Find the following limits.

(1). $\lim_{x\to 1} \frac{\sqrt{x^2+3}-2\sqrt{x}}{x^2-1}$

$$\lim_{x\to 1} \frac{\sqrt{x^2+3}-2\sqrt{x}}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x-3)}{(x-1)(x+1)(\sqrt{x^2+3}+2\sqrt{x})} = -\frac{1}{4}.$$

(2). $\lim_{x\to 2^-} \frac{x-2}{|x^2-5x+6|}$

$$\lim_{x\to 2^-} \frac{x-2}{|x^2-5x+6|} = \lim_{x\to 2^-} \frac{x-2}{|(x-2)(x-3)|} = \lim_{x\to 2^-} \frac{x-2}{(x-2)(x-3)} = -1.$$

4. (14 points) Suppose $f(x) = x$ for all $x \in [1, 2]$. Show that f is integrable on [1, 2] and find $\int_1^2 x$.

Proof For $n \in \mathbb{N}$, consider the partition for [1, 2]

$$P_n = \left\{1,\ 1+\frac{1}{n},\ 1+\frac{2}{n},\ \cdots, 1+\frac{n-1}{n},\ 2\right\},$$

in which $\Delta x_i = \frac{1}{n}$ for all $i = 1, \ldots, n$. Since $f(x) = x$ is increasing on [1, 2], we have $M_i = 1+\frac{i}{n}$ and $m_i = 1+\frac{i-1}{n}$ for each $i \in \{1, \ldots, n\}$. Thus

$$U(f, P_n) = \sum_{i=1}^{n} \left(1+\frac{i}{n}\right)\frac{1}{n} = 1 + \frac{1}{n^2}\sum_{i=1}^{n} i = 1 + \frac{n+1}{2n},$$

and

$$L(f, P_n) = \sum_{i=1}^{n} \left(1+\frac{i-1}{n}\right)\frac{1}{n} = 1 + \frac{1}{n^2}\sum_{i=1}^{n-1} i = 1 + \frac{n-1}{2n},$$

which imply $\lim_{n\to\infty} U(f, P_n) = \frac{3}{2}$ and $\lim_{n\to\infty} L(f, P_n) = \frac{3}{2}$. Therefore $U(f) \leq \frac{3}{2}$ and $L(f) \geq \frac{3}{2}$. But since $L(f) \leq U(f)$, we know that they have to be the same, which tells us that f is integrable on [1, 2] and $\int_1^2 x = U(f) = L(f) = \frac{3}{2}$. ☐

5. (10 points) Let f be continuous on $[a,\, b]$ and suppose that $f(x) \leq 0$ for all $x \in [a,\, b]$. Prove that if there exists a point $c \in [a,\, b]$ such that $f(c) < 0$, then $\int_a^b f < 0$.

Proof Since f is continuous at c and $f(c) < 0$, for $\varepsilon = \frac{-f(c)}{2} > 0$, $\exists \delta > 0$ such that whenever $x \in N(c, \delta) \cap [a,\, b]$, we have $|f(x) - f(c)| < \varepsilon = -\frac{f(c)}{2}$. Therefore, whenever $x \in N(c, \delta) \cap [a,\, b]$, $f(x) < f(c) + \varepsilon = \frac{f(c)}{2} < 0$. Note that $f(x) \leq 0$ for all $x \in [a,\, b]$, by Additivity, we have

$$\int_a^b f = \int_{[a,\, b] \cap N(c,\delta)} f + \int_{[a,\, b] \setminus N(c,\delta)} f \leq \int_{[a,\, b] \cap N(c,\delta)} \frac{f(c)}{2} < 0.$$

□