MA 452/502: Introduction to Real Analysis  
Solutions to Exam 2  

July 30, 2007  

There are five problems in this exam for a maximum of 100 points. Please show your work and present your solutions in a well organized way. No work, no credit.

1. (40 points) Mark each statement True or False. Justify each answer.

   (1). Let $f : D \to \mathbb{R}$ and $c \in D' \cap D$, then $f$ has a limit at $c$ iff $f$ is continuous at $c$.

   False. Let $f(x) = x$ for $x \neq 0$ and $f(0) = 1$, then $f(x)$ has a limit 0 at 0, but it is not continuous at 0.

   (2). Let $f : \mathbb{N} \to \mathbb{R}$, then $f$ is continuous on $\mathbb{N}$.

   True. That is because all the points in $\mathbb{N}$ are isolated points of $\mathbb{N}$.

   (3). Let $f : D \to \mathbb{R}$ and define $|f| : D \to \mathbb{R}$ by $|f|(x) = |f(x)|$, and $f^2 : D \to \mathbb{R}$ by $f^2(x) = (f(x))^2$. If $|f|$ is continuous at $c \in D$, then $f^2(x)$ is continuous at $c$.

   True. That is because of the product rule of continuous functions. Note that $f^2(x) = |f|(x) \cdot |f|(x)$, and $|f|(x)$ is continuous at $c$, we have $f^2(x)$ is continuous at $c$.

   (4). Every continuous function on a bounded interval is bounded.

   False. Let $f(x) = \frac{1}{x}$ for $x \in (0, 1)$, which is a continuous function on a bounded interval, but it is not bounded.

   (5). Equation $x^{2007} + 1 = 4x^{1006}$ has at least one real solution in $(0, 1)$.

   True. Define $f(x) = x^{2007} - 4x^{1006} + 1$, which is a polynomial function. Since $f(0) = 1 > 0$, $f(1) = -2 < 0$, by the Intermediate Value Theorem, $\exists c \in (0, 1)$ such that $f(c) = 0$, that is, $x^{2007} + 1 = 4x^{1006}$ has at least one real solution in $(0, 1)$.

   (6). Let $c$ be a point in the interval $I$, and suppose $f : I \to \mathbb{R}$. If $f'(c) > 0$, then $f$ is continuous at $c$.

   True. That is due to Theorem 25.6.

   (7). Let $c$ be a point in the interval $I$, and suppose $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$. If $f$ and $g$ are differentiable at $c$, then the composite function $g \circ f$ is differentiable at $c$.

   False. Let $f(x) = -x$, $g(x) = \sqrt{x}$ for $x \in (0, 1)$, then $f$ and $g$ are differentiable on $(0, 1)$, but $g \circ f$ is undefined on $(0, 1)$.
(8). Suppose \( f \) and \( g \) are continuous on \([0, 10]\) and differentiable on \((0, 10)\). If \( f'(x) = g'(x) \) for all \( x \in (0, 10) \) and \( f(\pi) = g(\pi) \), then \( f(x) = g(x) \).

**True.** Let \( h(x) = f(x) - g(x) \), then \( h \) is continuous on \([0, 10]\) and differentiable on \((0, 10)\), and \( h'(x) = 0 \) for all \( x \in (0, 10) \). By Theorem 26.6, \( h(x) = \text{Constant} \). Since \( h(\pi) = 0 = \text{Constant} \), we have \( h(x) = 0 \), and therefore \( f(x) = g(x) \).

(9). Let \( f : [a, b] \to \mathbb{R} \) be bounded and define \( |f| : [a, b] \to \mathbb{R} \) by \( |f|(x) = |f(x)| \). If \( |f| \) is integrable on \([a, b]\), then \( f \) is integrable on \([a, b]\).

**False.** Let \( f(x) = 1 \) if \( x \in \mathbb{Q} \cap [0, 1] \) and \( f(x) = -1 \) if \( x \in [0, 1] \setminus \mathbb{Q} \). Then \( f \) is not integrable on \([0, 1]\). But \( |f|(x) = 1 \) for all \( x \in [0, 1] \), which is integrable.

(10). If \( f \) is neither monotone nor continuous on \([a, b]\), then \( f \) is not integrable on \([a, b]\).

**False.** Let \( f(x) = 2 - x \) for \( x \in [0, 1] \) and \( f(x) = 4x + 3 \) for \( x \in (1, 2] \), then \( f \) is integrable on \([0, 2]\) but neither monotone nor continuous on \([0, 2]\).

2. (24 points)

(1). Prove the following statement by using only the definition:

\[
\lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0.
\]

**Proof** Clearly \( f(x) := x \sin \left( \frac{1}{x} \right) \) is well defined for all \( x \neq 0 \). For any \( \varepsilon > 0 \), \( \exists \delta = \varepsilon \) such that whenever \( 0 < |x| < \delta \), we have

\[
|x \sin \left( \frac{1}{x} \right) - 0| \leq |x| < \varepsilon.
\]

By Definition 20.1,

\[
\lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0.
\]

(2). Let

\[
f(x) = \begin{cases} 
2x + 5 & \text{if } x \leq 1, \\
9x^2 - 2 & \text{if } x > 1.
\end{cases}
\]

Prove that \( f(x) \) is continuous but NOT differentiable at 1.

**Proof** Clearly, \( f(x) \) is defined on \( \mathbb{R} \). Since

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 5) = 7, \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (9x^2 - 2) = 7,
\]

we have \( \lim_{x \to 1} f(x) = f(1) = 7 \). By Theorem 21.2, \( f(x) \) is continuous at 1.
To see that \( f(x) \) is not differentiable at 1, we compute the left and right derivatives of \( f(x) \) at 1. Since

\[
f'_-(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{2x - 2}{x - 1} = 2, \quad f'_+(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{9x^2 - 9}{x - 1} = 18,
\]
we have \( f'_-(1) \neq f'_+(1) \), which implies \( f \) is not differentiable at 1. \( \Box \)

3. Let \( f(x) = \sqrt{2x - 1} \), find \( f'(2) \) by using only the definition.

**Solution**

\[
f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\sqrt{2x - 1} - \sqrt{3}}{x - 2} = \lim_{x \to 2} \frac{2(x - 2)}{(x - 2)(\sqrt{2x - 1} + \sqrt{3})} = \frac{1}{\sqrt{3}}.
\]

3. (12 points) Find the following limits.

1. \( \lim_{x \to 1} \frac{\sqrt{x^2 + 3 - 2\sqrt{x}}}{x - 1} \)

\[
\lim_{x \to 1} \frac{\sqrt{x^2 + 3 - 2\sqrt{x}}}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x - 3)}{(x - 1)(x + 1)(\sqrt{x^2 + 3 + 2\sqrt{x}})} = \frac{1}{4}.
\]

2. \( \lim_{x \to 2^-} \frac{x - 2}{|x^2 - 5x + 6|} \)

\[
\lim_{x \to 2^-} \frac{x - 2}{|x^2 - 5x + 6|} = \lim_{x \to 2^-} \frac{x - 2}{|(x - 2)(x - 3)|} = \lim_{x \to 2^-} \frac{x - 2}{(x - 2)(x - 3)} = -1.
\]

4. (14 points) Suppose \( f(x) = x \) for all \( x \in [1, 2] \). Show that \( f \) is integrable on \([1, 2] \) and find \( \int_1^2 x \).

**Proof** For \( n \in \mathbb{N} \), consider the partition for \([1, 2] \)

\[
P_n = \left\{ 1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, \ldots, 1 + \frac{n - 1}{n}, 2 \right\},
\]
in which \( \Delta x_i = \frac{1}{n} \) for all \( i = 1, \ldots, n \). Since \( f(x) = x \) is increasing on \([1, 2] \), we have \( M_i = 1 + \frac{i}{n} \) and \( m_i = 1 + \frac{i - 1}{n} \) for each \( i \in \{1, \ldots, n\} \). Thus

\[
U(f, P_n) = \sum_{i=1}^{n} \left( 1 + \frac{i}{n} \right) \frac{1}{n} = 1 + \frac{1}{n^2} \sum_{i=1}^{n} i = 1 + \frac{n + 1}{2n},
\]
and

\[
L(f, P_n) = \sum_{i=1}^{n} \left( 1 + \frac{i - 1}{n} \right) \frac{1}{n} = 1 + \frac{1}{n^2} \sum_{i=1}^{n-1} i = 1 + \frac{n - 1}{2n},
\]
which imply \( \lim_{n \to \infty} U(f, P_n) = \frac{3}{2} \) and \( \lim_{n \to \infty} L(f, P_n) = \frac{3}{2} \). Therefore \( U(f) \leq \frac{3}{2} \) and \( L(f) \geq \frac{3}{2} \). But since \( L(f) \leq U(f) \), we know that they have to be the same, which tells us that \( f \) is integrable on \([1, 2] \) and \( \int_1^2 x = U(f) = L(f) = \frac{3}{2} \). \( \Box \)
5. (10 points) Let \( f \) be continuous on \([a, b]\) and suppose that \( f(x) \leq 0 \) for all \( x \in [a, b] \). Prove that if there exists a point \( c \in [a, b] \) such that \( f(c) < 0 \), then \( \int_a^b f < 0 \).

**Proof** Since \( f \) is continuous at \( c \) and \( f(c) < 0 \), for \( \varepsilon = \frac{-f(c)}{2} > 0 \), \( \exists \delta > 0 \) such that whenever \( x \in N(c, \delta) \cap [a, b] \), we have \(|f(x) - f(c)| < \varepsilon = \frac{-f(c)}{2}\). Therefore, whenever \( x \in N(c, \delta) \cap [a, b] \), \( f(x) < f(c) + \varepsilon = \frac{f(c)}{2} < 0 \). Note that \( f(x) \leq 0 \) for all \( x \in [a, b] \), by Additivity, we have

\[
\int_a^b f = \int_{[a, b] \cap N(c, \delta)} f + \int_{[a, b] \setminus N(c, \delta)} f \leq \int_{[a, b] \cap N(c, \delta)} \frac{f(c)}{2} < 0.
\]

\( \square \)