MA 585: Probability
Solutions to Exam 2

April 14, 2008

There are five problems for a maximum of 50 points. Please show your work in a well organized way. No work, no credit.

1. (10 points) Suppose that the number of customers entering a post office between 9:00am and 10:00am follows a Poisson(λ) and that with probability 1/3 an incoming customer is a male. Let X be the number of male customers entering the office during the time period. Find $\mathbb{P}\{X = 2\}$.

Solution. Let Y be the number of customers entering the post office. Then,

$$\begin{aligned}
\mathbb{P}\{X = 2\} &= \sum_{n=2}^{\infty} \mathbb{P}\{X = 2|Y = n\}\mathbb{P}\{Y = n\} \\
&= \sum_{n=2}^{\infty} \binom{n}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{n-2} \frac{\lambda^n}{n!} e^{-\lambda} \\
&= \frac{\lambda^2 e^{-\lambda}}{18} \sum_{n=2}^{\infty} \frac{\left(\frac{2\lambda}{3}\right)^{n-2}}{(n-2)!} = \frac{\lambda^2 e^{-\lambda}}{18} \sum_{k=0}^{\infty} \frac{\left(\frac{2\lambda}{3}\right)^{k}}{k!} \\
&= \frac{\lambda^2 e^{-\lambda}}{18} e^{\frac{2\lambda}{3}} = \frac{\lambda^2}{18} e^{-\frac{\lambda}{3}},
\end{aligned}$$

where the second equality is because the conditional probability of X given $Y = n$ is Binomial$(n, 1/3)$.
Remark: In fact, we can derive that X has a Poisson $\left(\frac{\lambda}{3}\right)$. This is called the thinning property of a Poisson random variable.

2. (10 points) Let X_1, X_2, X_3, X_4, X_5 be five i.i.d continuous random variables with common cdf function F. Define $X_1^* = \min\{X_1, X_2, X_3, X_4, X_5\}$.

(1). Find the cdf of X_1^* (in terms of F).

(2). Find $\mathbb{P}\{X_1^* = X_3\}$.

Solution.

(1). Denote the cdf of X_1^* by $F_*(x)$, then

$$\begin{aligned} F_*(x) &= \mathbb{P}\{X_1^* \le x\} = 1 - \mathbb{P}\{X_1^* > x\} \\ &= 1 - \mathbb{P}\{X_1 > x,\, X_2 > x,\, X_3 > x,\, X_4 > x,\, X_5 > x\} = 1 - \prod_{j=1}^{5} \mathbb{P}\{X_j > x\} \\ &= 1 - \big(1 - F(x)\big)^5. \end{aligned}$$

(2). $\mathbb{P}\{X_1^* = X_3\} = \mathbb{P}\{X_3 \text{ is the smallest among } X_1, \dots, X_5\} = \frac{1}{5}$. [The conclusion follows from Problem 59 of Ch 2.]

3. (10 points) Prove that

$$\lim_{n\to\infty} 2^{1-2n} \sum_{k=0}^{n} \binom{2n}{k} = 1.$$

Proof Let $\{X_i\}$ be a sequence of i.i.d Bernoulli$\left(\frac{1}{2}\right)$ random variables. Clearly $\mathbb{E}X_i = \frac{1}{2}$, $\mathrm{Var}(X_i) = \frac{1}{4}$. By the Central Limit Theorem, we have that for $a \in \mathbb{R}$

$$\lim_{k\to\infty} \mathbb{P}\left\{ \frac{\sum_{i=1}^{k} X_i - \frac{k}{2}}{\sqrt{\frac{k}{4}}} \le a \right\} = \Phi(a).$$

Especially,

$$\lim_{n\to\infty} \mathbb{P}\left\{ \frac{\sum_{i=1}^{2n} X_i - n}{\sqrt{\frac{n}{2}}} \le 0 \right\} = \lim_{n\to\infty} \mathbb{P}\left\{ \sum_{i=1}^{2n} X_i \le n \right\} = \Phi(0) = \frac{1}{2}.$$

Notice that $\sum_{i=1}^{2n} X_i \sim$Bin$(2n, 1/2)$, we have

$$\lim_{n\to\infty} \sum_{i=0}^{n} \binom{2n}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{2n-i} = \frac{1}{2},$$

and the conclusion follows immediately. □

4. (10 points) Let $Z \sim N(0, 1)$. Prove that for $x \ge 0$

$$\mathbb{P}\{Z \ge x\} \le e^{-x^2/2}.$$

[Hint: One way to solve the problem is to apply Markov inequality to $Y = e^{Zt}$, and to minimize the upper bound with respect to t.]

Proof Notice that, by Markov inequality, for $t \geq 0$ we have

$$\mathbb{P}\{Z \geq x\} = \mathbb{P}\left\{e^{Zt} \geq e^{xt}\right\} \leq \frac{\mathbb{E}[e^{Zt}]}{e^{xt}} = e^{\frac{t^2}{2} - xt} = e^{\frac{1}{2}(t-x)^2 - \frac{x^2}{2}}.$$

Choosing $t = x$, we prove

$$\mathbb{P}\{Z \geq x\} \leq e^{-x^2/2}.$$

□

5. (10 points) Suppose the joint density function of X and Y is given by

$$f(x,y) = \frac{y^2 - x^2}{8} e^{-y}, \quad 0 < y < \infty, \ |x| \leq y.$$

Find $\mathbb{E}[X]$ and $\mathbb{E}[X|Y = y]$.
Solution. Notice that for $y > 0$

$$f_Y(y) = \int_{-y}^{y} \frac{y^2 - x^2}{8} e^{-y} dx = \frac{4}{3} y^3 e^{-y},$$

and

$$f_{X|Y}(x|y) = \frac{3(y^2 - x^2)}{4y^3}, \quad |x| \leq y,$$

we have

$$\mathbb{E}[X|Y = y] = \int_{-y}^{y} x \frac{3(y^2 - x^2)}{4y^3} dx = 0.$$

Therefore, by the conditioning argument,

$$\mathbb{E}[X] = \int_{0}^{\infty} \mathbb{E}[X|Y = y] f_Y(y) dy = 0.$$