MA 452/502: Introduction to Real Analysis
Final Exam

Name:______________________________________ Score: ____________/100

December 03, 2007

There are 7 problems in this exam for a maximum of 100 points. Please show your work and present your solutions in a well organized way. No work, no credit.

1. (25 points) Disprove the following statements by constructing counterexamples. Justify your answer please.

(1). If a nonempty bounded set $S \subseteq \mathbb{R}$ contains its maximum and minimum, then S is compact.

(2). Every unbounded sequence has no convergent subsequences.

(3). If $\lim_{x\to c} f^2(x) = L > 0$, then $\lim_{x\to c} f(x)$ exists and is finite.

(4). Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$. If f and g are uniformly continuous on I, then the product function fg is uniformly continuous on I.

(5). Let $f(x)$ and $g(x)$ be integrable on $[a, b]$, and let $f(x) \le h(x) \le g(x)$ for all $x \in [a, b]$, then $h(x)$ is integrable on $[a, b]$.

2. (9 points) Let $S \subseteq \mathbb{R}$ and $s \in S'$, prove that either $s \in \text{int}S$ or $s \in \text{bd}S$.

3. (16 points) Prove each of the following statements by using only the definition.

(1). Let $s_n = \sqrt{n^2 + 2} - n$, prove that (s_n) converges to 0.

(2). Prove $f(x) = |x - 2| + x^2$ is continuous at 2.

4. (14 points) Let $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2 + s_n}$ for $n \geq 1$. Prove that $\big(s_n\big)$ is Cauchy, and find $\lim_{n\to\infty} s_n$.

5. (10 points) Let $f : [a, b] \to [a, b]$ be a continuous function, prove that $f(x)$ has a *fixed point.* That is, prove that $\exists\, c \in [a, b]$ such that $f(c) = c$.

6. (14 points) Let

$$f(x) = \begin{cases} ax + 5 & \text{if } x \leq 1, \\ 3x^2 + b & \text{if } x > 1, \end{cases}$$

where $a,\ b \in \mathbb{R}$ are unknown constants. Please determine a and b such that $f(x)$ is differentiable on $\mathbb{R}$.

7. (12 points) Let $f(x)$ be continuous on $\mathbb{R}$, prove that for any $[a,\ b] \subset \mathbb{R}$,

$$\lim_{h \to 0} \int_a^b |f(x+h) - f(x)| = 0.$$