MA 452/502: Introduction to Real Analysis
Final Exam

August 1, 2007

There are six problems in this exam for a maximum of 100 points. Please show your work and present your solutions in a well organized way. No work, no credit.

1. (40 points) Mark each statement True or False. Justify each answer.

(1). Suppose x and y are two positive irrational numbers, then $x + y$ is irrational.

(2). Every infinite set has at least one accumulation point.

(3). If a sequence converges to a positive number, then there are at most finitely many non-positive terms in the sequence.

(4). Every monotone sequence has a limit.

(5). Every bounded sequence has a Cauchy subsequence.

(6). Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \frac{2n^2-1}{n^3}$, $n \in \mathbb{N}$, then f has a limit at 5.

(7). Let $f \cdot g$ and g be continuous functions on $[0, 1]$, then f is continuous on $[0, 1]$.

(8). Suppose f is continuous on $[0, 5]$, and differentiable on $(0, 5)$. If $f'(3) = 0$, then there exists $N(3, \delta)$, a δ-neighborhood of 3, such that $f(x) = Constant$ for all $x \in N(3, \delta)$.

(9). If a bounded function f has infinitely many discontinuous points on $[a, b]$, then it is not Riemann integrable.

(10). Let f be monotone on $[a, b]$, g be continuous on $[c, d]$ with $f\big([a, b]\big) \subseteq [c, d]$ then $g \circ f$ is integrable on $[a, b]$.

2. (8 points) Prove that a boundary point of a set S is either an accumulation point of S or an isolated point of S.

3. (15 points) Prove the following statement by using only the corresponding definitions.

(1). $\lim_{n\to\infty} \frac{3n-4}{2n+5} = \frac{3}{2}$.

(2). Let $f(x) = x^2 \cos\left(\frac{1}{x}\right)$ for $x \neq 0$ and f(0)=0. Then $f(x)$ is differentiable at 0.

4. (15 points) Find the following limits.

(1). $\lim_{n\to\infty} \frac{2^n}{n^2-5n-6}$

(2). $\lim_{n\to\infty} \frac{\sqrt{n^2+3}-2\sqrt{n}}{n^2-1}$

(3). $\lim_{x\to-2^+} \frac{x^2-4}{|x^2-x-6|}$

5. (10 points) Let

$$f(x) = \{ \begin{array}{ll} k_1x-5 & \text{if } x<2 \\ 3-k_2x^2 & \text{if } x\geq 2 \end{array}$$

Please determine k_1 and k_2, which are real numbers, such that $f(x)$ is differentiable at 2.

6. (12 points) Let f and g be integrable on $[a, b]$ and suppose that h is defined on $[a, b]$ such that $f(x) \leq h(x) \leq g(x)$ for all $x \in [a, b]$. Prove that if $\int_a^b f = \int_a^b g$, then h is integrable on $[a, b]$. Find an example to show that the condition $\int_a^b f = \int_a^b g$ is necessary.